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Aim and Scope

International Journal of Neutrosophic Science (IJNS) is a peer-review journal publishing high quality experimental and theoretical research in all areas of Neutrosophic and its Applications. IJNS is published quarterly. IJNS is devoted to the publication of peer-reviewed original research papers lying in the domain of neutrosophic sets and systems. Papers submitted for possible publication may concern with foundations, neutrosophic logic and mathematical structures in the neutrosophic setting. Besides providing emphasis on topics like artificial intelligence, pattern recognition, image processing, robotics, decision making, data analysis, data mining, applications of neutrosophic mathematical theories contributing to economics, finance, management, industries, electronics, and communications are promoted. Variants of neutrosophic sets including refined neutrosophic set (RNS). Articles evolving algorithms making computational work handy are welcome.

Topics of Interest

IJNS promotes research and reflects the most recent advances of neutrosophic Sciences in diverse disciplines, with emphasis on the following aspects, but certainly not limited to:

- | | |
|---|--|
| <input type="checkbox"/> Neutrosophic sets | <input type="checkbox"/> Neutrosophic algebra |
| <input type="checkbox"/> Neutrosophic topolog | <input type="checkbox"/> Neutrosophic graphs |
| <input type="checkbox"/> Neutrosophic probabilities | <input type="checkbox"/> Neutrosophic tools for decision making |
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| <input type="checkbox"/> Neutrosophic numerical measures | <input type="checkbox"/> Classical neutrosophic numbers |
| <input type="checkbox"/> A neutrosophic hypothesis | <input type="checkbox"/> The neutrosophic level of significance |
| <input type="checkbox"/> The neutrosophic confidence interval | <input type="checkbox"/> The neutrosophic central limit theorem |
| <input type="checkbox"/> Neutrosophic theory in bioinformatics | |
| <input type="checkbox"/> and medical analytics | <input type="checkbox"/> Neutrosophic tools for big data analytics |
| <input type="checkbox"/> Neutrosophic tools for deep learning | <input type="checkbox"/> Neutrosophic tools for data visualization |
| <input type="checkbox"/> Quadripartitioned single-valued | |
| <input type="checkbox"/> neutrosophic sets | <input type="checkbox"/> Refined single-valued neutrosophic sets |

- ☐ Applications of neutrosophic logic in image processing
- ☐ Neutrosophic logic for feature learning, classification, regression, and clustering
- ☐ Neutrosophic knowledge retrieval of medical images
- ☐ Neutrosophic set theory for large-scale image and multimedia processing
- ☐ Neutrosophic set theory for brain-machine interfaces and medical signal analysis
- ☐ Applications of neutrosophic theory in large-scale healthcare data
- ☐ Neutrosophic set-based multimodal sensor data
- ☐ Neutrosophic set-based array processing and analysis
- ☐ Wireless sensor networks Neutrosophic set-based Crowd-sourcing
- ☐ Neutrosophic set-based heterogeneous data mining
- ☐ Neutrosophic in Virtual Reality
- ☐ Neutrosophic and Plithogenic theories in Humanities and Social Sciences
- ☐ Neutrosophic and Plithogenic theories in decision making
- ☐ Neutrosophic in Astronomy and Space Sciences



On Neutro-*BE*-algebras and Anti-*BE*-algebras

(revisited)

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Abstract

In this paper, the concepts of Neutro-*BE*-algebra and Anti-*BE*-algebra are introduced, and some related properties and four theorems are investigated. We show that the classes of Neutro-*BE*-algebra and Anti-*BE*-algebras are alternatives of the class of *BE*-algebras.

Keywords: *BE*-algebra; Neutro-sophication; Neutro-*BE*-algebra; Anti-sophication; Anti-*BE*-algebra.

1. Introduction

Neutrosophy, introduced by F. Smarandache in 1998, is a new branch of philosophy that generalized the dialectics and took into consideration not only the dynamics of opposites, but the dynamics of opposites and their neutrals [8]. Neutrosophic Logic / Set / Probability / Statistics / Measure / Algebraic Structures etc. are all based on it. One of the most striking trends in the neutrosophic theory is the hybridization of neutrosophic set with other potential sets such as rough set, bipolar set, soft set, vague set, etc. The different hybrid structures such as rough neutrosophic set, single valued neutrosophic rough set, bipolar neutrosophic set, single valued neutrosophic vague set, etc. are proposed in the literature in a short period of time. Neutrosophic set has been a very important tool in all various areas of data mining, decision making, e-learning, engineering, computer science, graph theory, medical diagnosis, probability theory, topology, social science, etc [9-13].

A classical Algebra may be transformed into a NeutroAlgebra by a process called neutro-sophication, and into an AntiAlgebra by a process called anti-sophication.

In [2], H.S. Kim et al. introduced the notion of a *BE*-algebra as a generalization of a *BCK*-algebra. S.S. Ahn et al. introduced the notion of ideals in *BE*-algebras, and they stated and proved several properties of such ideals [1]. A. Borumand Saeid et al defined some filters in *BE*-algebras and investigated relation between them [3]. A. Rezaei et al. investigated the relationship between Hilbert algebras and *BE*-algebras and showed that commutative self-distributive *BE*-algebras and Hilbert algebras are equivalent [4]. In this paper, the concepts of a Neutro-*BE*-algebra and Anti-*BE*-algebra are introduced, and some related properties are investigated. We show that the class of Neutro-*BE*-algebra is an alternative of the class of *BE*-algebras.

2. NeutroLaw, NeutroOperation, NeutroAxiom, and NeutroAlgebra

In this section, we review the basic definitions and some elementary aspects that are necessary for this paper.

The Neutrosophy's Triplet is $\langle A \rangle, \langle \text{neutro}A \rangle, \langle \text{anti}A \rangle$, where $\langle A \rangle$ may be an item (concept, idea, proposition, theory, structure, algebra, etc.), $\langle \text{anti}A \rangle$ the opposite of $\langle A \rangle$, while $\langle \text{neutro}A \rangle$ {also the notation $\langle \text{neut}A \rangle$ was employed before} the neutral between these opposites. Based on the above triplet the following Neutrosophic Principle one has: a law of composition defined on a given set may be true (T) for some set elements, indeterminate (I) for other set's elements, and false (F) for the remainder of the set's elements; we call it NeutroLaw. A law of composition defined on a given sets, such that the law is false (F) for all set's elements is called AntiLaw. Similarly, an operation defined on a given set may be well-defined for some set elements, indeterminate for other set's elements, and undefined for the remainder of the set's elements; we call it NeutroOperation. While, an operation defined on a given set that is undefined for all set's elements is called AntiOperation.

In classical algebraic structures, the laws of compositions or operations defined on a given set are automatically well-defined [i.e. true (T) for all set's elements], but this is idealistic. Consequently, an axiom (let's say Commutativity, or Associativity, etc.) defined on a given set, may be true (T) for some set's elements, indeterminate (I) for other set's elements, and false (F) for the remainder of the set's elements; we call it NeutroAxiom. In classical algebraic structures, similarly an axiom defined on a given set is automatically true (T) for all set's elements, but this is idealistic too. A NeutroAlgebra is a set endowed with some NeutroLaw (NeutroOperation) or some NeutroAxiom. The NeutroLaw, NeutroOperation, NeutroAxiom, NeutroAlgebra and respectively AntiLaw, AntiOperation, AntiAxiom and AntiAlgebra were introduced by Smarandache in 2019 [6] and afterwards he recalled, improved and extended them in 2020 [7]. Recently, the concept of a Neutrosophic Triplet of BI -algebra was defined [5].

3. Neutro-BE-algebras, Anti-BE-Algebras

Definition 3.1. (Definition of classical BE-algebras [1])

An algebra $(X, *, 0)$ of type $(2, 0)$ (i.e. X is a nonempty set, $*$ is a binary operation and 0 is a constant element of X) is said to be a *BE-algebra* if:

(L) **The law $*$ is well-defined**, i.e. $(\forall x, y \in X)(x * y \in X)$.

And the following axioms are totally true on X :

$$(BE1) (\forall x \in X)(x * x = 0),$$

$$(BE2) (\forall x \in X)(0 * x = x),$$

$$(BE3) (\forall x \in X)(x * 0 = 0),$$

$$(BE4) (\forall x, y, z \in X, \text{ with } x \neq y)(x * (y * z) = y * (x * z)).$$

Example 3.2.

(i) Let \mathbb{N} be the set of all natural numbers and $*$ be the binary operation on \mathbb{N} defined by

$$x * y = \begin{cases} y & \text{if } x = 1; \\ 1 & \text{if } x \neq 1. \end{cases}$$

Then $(\mathbb{N}, *, 1)$ is a BE-algebra.

(ii) Let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and let $*$ be the binary operation on \mathbb{N}_0 defined by

$$x * y = \begin{cases} 0 & \text{if } x \geq y; \\ y - x & \text{otherwise.} \end{cases}$$

Then $(\mathbb{N}_0, *, 0)$ is a BE-algebra.

Definition 3.3. (Neutro-sophications)

The Neutro-sophication of the Law (degree of well-defined, degree of indeterminacy, degree of outer-defined)

(NL) $(\exists x, y \in X)(x * y \in X)$ and $(\exists x, y \in X)(x * y = \text{indeterminate or } x * y \notin X)$,

The Neutro-sophication of the Axioms (degree of truth, degree of indeterminacy, degree of falsehood)

(NBE1) $(\exists x \in X)(x * x = 0)$ and $(\exists x \in X)(x * x = \text{indeterminate or } x * x \neq 0)$,

(NBE2) $(\exists x \in X)(0 * x = x)$ and $(\exists x \in X)(0 * x = \text{indeterminate or } 0 * x \neq x)$,

(NBE3) $(\exists x \in X)(x * 0 = 0)$ and $(\exists x \in X)(x * 0 = \text{indeterminate or } x * 0 \neq 0)$,

(NBE4) $(\exists x, y, z \in X, \text{ with } x \neq y)(x * (y * z) = y * (x * z))$ and

$(\exists x, y, z \in X, \text{ with } x \neq y)(x * (y * z) = \text{indeterminate or } x * (y * z) \neq y * (x * z))$.

Definition 3.4. (Anti-sophications)

The Anti-sophication of the Law (totally outer-defined)

(AL) $(\forall x, y \in X)(x * y \notin X)$.

The Anti-sophication of the Axioms (totally false)

(ABE1) $(\forall x \in X)(x * x \neq 0)$,

(ABE2) $(\forall x \in X)(0 * x \neq x)$,

(ABE3) $(\forall x \in X)(x * 0 \neq 0)$,

(ABE4) $(\forall x, y, z \in X, \text{ with } x \neq y)(x * (y * z) \neq y * (x * z))$.

Definition 3.5. (Neutro-BE-algebras)

A Neutro-BE-algebra is an alternative of BE-algebra that has at least a (NL) or at least one (NBE i), $i \in \{1, 2, 3, 4\}$, with no anti-law and no anti-axiom.

Example 3.6.

(i) Let \mathbb{N} be the set of all natural numbers and $*$ be the Neutro-sophication of the Law $*$ on \mathbb{N} from Example 2.2. (i) defined by

$$x * y = \begin{cases} y & \text{if } x = 1; \\ \frac{1}{2} & \text{if } x \in \{3, 5, 7\}; \\ 1 & \text{otherwise.} \end{cases}$$

Then $(\mathbb{N}, *, 1)$ is a Neutro-BE-algebra. Since

(NL) if $x \in \{3, 5, 7\}$, then $x * y = \frac{1}{2} \notin \mathbb{N}$, for all $y \in \mathbb{N}$, while if $x \notin \{3, 5, 7\}$ and $x \in \mathbb{N}$, then $x * y \in \{1, y\} \subseteq \mathbb{N}$, for all $y \in \mathbb{N}$.

(NBE1) $1 * 1 = 1 \in \mathbb{N}$ and $3 * 3 = \frac{1}{2} \notin \mathbb{N}$,

(BE2) holds always since $1 * x = x$, for all $x \in \mathbb{N}$.

(NBE3) $5 * 1 = \frac{1}{2} \neq 1$ and if $x \notin \{3, 5, 7\}$, then $x * 1 = 1$,

(NBE4) $5 * (3 * 4) = 5 * \frac{1}{2} = ?$ (*indeterminate*) and $3 * (5 * 4) = 3 * \frac{1}{2} = ?$ (*indeterminate*)

Also, $2 * (3 * 4) = 2 * \frac{1}{2} = ?$ (*indeterminate*), but $3 * (2 * 4) = 3 * 1 = \frac{1}{2}$.

Further, $4 * (8 * 2) = 4 * 1 = 1 = 8 * (4 * 2)$.

(ii) Let S be a nonempty set and $\mathcal{P}(S)$ be the power set of S . Then $(\mathcal{P}(S), \cap, \emptyset)$ is a Neutro-BE-algebra.

\cap is the binary set intersection operation, but

(NBE1) is valid, since $\emptyset \cap \emptyset = \emptyset$ and for all $\emptyset \neq A \in \mathcal{P}(S)$, $A \cap A = A \neq \emptyset$.

(NBE2) $\emptyset \cap \emptyset = \emptyset$ and if $\emptyset \neq A$, then $\emptyset \cap A = \emptyset \neq A$,

(BE3) holds, since $A \cap \emptyset = \emptyset$,

(BE4) holds, since $A \cap (B \cap C) = B \cap (A \cap C)$.

(iii) Similarly, $(\mathcal{P}(S), \cup, \emptyset)$, $(\mathcal{P}(S), \cap, S)$, $(\mathcal{P}(S), \cup, S)$, where \cup is the binary set union operation, are Neutro-BE-algebras.

(iv) Let $X := \{0, a, b, c, d\}$ be a set with the following table.

Table 1

$*$	0	a	b	c	d
0	c	a	b	c	a
a	b	0	b	c	d
b	0	a	0	c	c
c	$?$	0	b	0	b
d	0	0	0	0	$?$

Then $(X, *, 0)$ is a Neutro-BE-algebra.

(NL) $c * 0 = ?$ (*indeterminate*), and $d * d = ?$ (*indeterminate*), and for all $x, y \in \{0, a, b\}$, then $x * y \in X$.

(NBE1) $a * a = 0$ and $0 * 0 = c \neq 0$ or $d * d = ?$ (*indeterminate*).

(NBE2) holds since $0 * b = b$, and $0 * d = a \neq d$.

(NBE3) $c * 0 = ?$ (indeterminate) $\neq 0$ and if $x \in \{b, d\}$, then $x * 0 = 0$,

(NBE4) $d * (c * b) = d * b = 0 \neq c * (d * b) = c * 0 = ?$ (indeterminate) and

$a * (b * c) = a * c = c = b * (a * c)$.

(v) Let S be a nonempty set and $\mathcal{P}(S)$ be the power set of S . Then $(\mathcal{P}(S), -, \emptyset)$ is an Anti-BE-algebra, where $-$ is the binary operation of set subtraction, because:

(BE1) is valid, since $A - A = \emptyset$,

(NBE2) holds, since $\emptyset - A = \emptyset \neq A$ and $\emptyset - \emptyset = \emptyset$,

(NBE3) holds, since $A - \emptyset = A \neq \emptyset$ and $\emptyset - \emptyset = \emptyset$

(ABE4) is valid, since for $A \neq B$, one has $A - (B - C) \neq B - (A - C)$, because:

$x \in A - (B - C)$ means $(x \in A \text{ and } x \notin B - C)$, or $\{x \in A \text{ and } (x \notin B \text{ or } x \in C)\}$, or $\{(x \in A \text{ and } x \notin B) \text{ or } (x \in A \text{ and } x \in C)\}$; while $x \in B - (A - C)$ means $\{(x \in B \text{ and } x \notin A) \text{ or } (x \in B \text{ and } x \in C)\}$.

(vi) Let \mathbb{R} be the set of all real numbers and $*$ be a binary operation on \mathbb{R} defined by $x * y = |x - y|$. Then $(\mathbb{R}, *, 0)$ is a Neutro-BE-algebra.

(BE1) holds, since $x * x = |x - x| = 0$, for all $x \in \mathbb{R}$.

(NBE2) is valid, since if $x \geq 0$, then $x * 0 = |x - 0| = |x| = x$, and if $x < 0$, then $x * 0 = |x - 0| = |x| = -x \neq x$.

(NBE3) is valid, since if $x \neq 0$, then $0 * x = |0 - x| = |-x| \neq 0$, and if $x = 0$, then $0 * 0 = 0$.

(NBE4) holds, if $x = 2, y = 3, z = 4$ we get $|2 - |3 - 4|| = |2 - 1| = 1$ and $|3 - |2 - 4|| = |3 - 2| = 1$;

while for $x = 4, y = 8, z = 3$ we get $|4 - |8 - 3|| = |4 - 5| = 1$ and $|8 - |4 - 3|| = |8 - 1| = 7 \neq 1$.

Theorem 3.7.

The total number of Neutro-BE-algebras is 31.

Proof.

The classical BE-algebra has: 1 classical Law and 4 classical Axioms:

$1 + 4 = 5$ classical mathematical propositions.

Let C_n^m mean combinations of n elements taken by m , where n, m are positive integers, $n \geq m \geq 0$.

We transform (neutro-sophicate) the classical BE-algebra, by neutro-sophicating some of the 5 classical mathematical propositions, while the others remain classical (unchanged) mathematical propositions:

either only 1 of the 5 classical mathematical propositions (hence we have $C_5^1 = 5$ possibilities) – so 4 classical mathematical propositions remain unchanged,

or only 2 of the 5 classical mathematical propositions (hence we have $C_5^2 = 10$ possibilities) – so 3 classical mathematical propositions remain unchanged,

or only 3 of the 5 classical mathematical propositions (hence we have $C_5^3 = 10$ possibilities) – so 2 classical mathematical propositions remain unchanged,

or only 4 of the 5 classical mathematical propositions (hence we have $C_5^4 = 5$ possibilities) – so 1 classical mathematical proposition remains unchanged,

or all 5 of the 5 classical mathematical propositions (hence we have $C_5^1 = 1$ possibilities).

Whence the total number of possibilities will be:

$$C_5^1 + C_5^2 + C_5^3 + C_5^4 + C_5^5 = (1 + 1)^5 - C_5^0 = 2^5 - 1 = 31.$$

Definition 3.8. (Anti-BE-algebras)

An Anti-BE-algebra is an alternative of BE-algebra that has at least an (AL) or at least one (ABE i), $i \in \{1, 2, 3, 4\}$.

Example 3.9.

(i) Let \mathbb{N} be the natural number set and $X = \mathbb{N} \cup \{0\}$. Define a binary operation $*$ on X by $x *_A y = x^2 + y^2 + 1$. Then $(X, *, 0)$ is not a BE-algebra, nor a Neutro-BE-algebra, but an Anti-BE-algebra.

Since $x *_A x = x^2 + x^2 + 1 \neq 0$, for all $x \in X$, and so (ABE1) holds.

For all $x \in \mathbb{N}$, we have $x * 0 = x^2 + 1 \neq 0$, so (ABE2) is valid. By a similar argument (ABE3) is valid.

Since for $x \neq y$, one has $x *_A (y *_A z) = x^2 + (y^2 + z^2 + 1)^2 + 1 \neq y *_A (x *_A z) = y^2 + (x^2 + z^2 + 1)^2 + 1$,

thus (ABE4) is valid.

(ii) Let S be a nonempty set and $\mathcal{P}(S)$ be the power set of S . Define the binary operation Δ (i.e. symmetric difference) by $A \Delta B = (A \cup B) - (A \cap B)$ for every $A, B \in \mathcal{P}(S)$. Then $(\mathcal{P}(S), \Delta, S)$ is not a BE-algebra, nor Neutro-BE-algebra, but it is an Anti-BE-algebra.

Since $A \Delta A = \emptyset \neq S$ for every $A \in \mathcal{P}(S)$ we get (ABE1) holds, and so (BE1) and (NBE1) are not valid.

Also, for all $A, B, C \in \mathcal{P}(S)$ one has $A \Delta (B \Delta C) = B \Delta (A \Delta C)$. Thus, (BE4) is valid.

Since there is at least one anti-axiom (ABE1), then $(\mathcal{P}(S), \Delta, S)$ is an Anti-BE-algebra.

(iii) Let $\mathcal{U} = \{0, a, b, c, d\}$ be a universe of discourse, and a subset $S = \{0, c\}$, and the below binary well-defined Law $*$ with the following Cayley table.

Table 2

*	0	c
0	c	0
c	c	c

Then $(S, *, 0)$ is an Anti-BE-algebra, since (ABE1) is valid, because: $0 * 0 = c \neq 0$ and $c * c = c \neq 0$, and it is sufficient to have a single anti-axiom.

Theorem 3.10.

The total number of Anti-BE-algebras is 211.

Proof.

The classical *BE*-algebra has: 1 classical Law and 4 classical Axioms:

$1 + 4 = 5$ classical mathematical propositions.

Let C_n^m mean combinations of n elements taken by m , where n, m are positive integers, $n \geq m \geq 0$.

We transform (anti-sophicate) the classical *BE*-algebra, by anti-sophicating some of the 5 classical mathematical propositions, while the others remain classical (unchanged) or neutro-mathematical propositions:

either only 1 of the 5 classical mathematical propositions (hence we have $C_5^1 = 5$ subpossibilities) – so 4 classical mathematical propositions remain some unchanged others neutro-sophicated or $2^4 = 16$ subpossibilities; hence total number of possibilities in this case is: $5 \cdot 16 = 80$;

or 2 of the 5 classical mathematical propositions (hence we have $C_5^2 = 10$ subpossibilities) – so 3 classical mathematical propositions remain some unchanged other neutro-sophicated or $2^3 = 8$ subpossibilities; hence total number of possibilities in this case is: $10 \cdot 8 = 80$;

or 3 of the 5 classical mathematical propositions (hence we have $C_5^3 = 10$ subpossibilities) – so 2 classical mathematical propositions remain some unchanged other neutro-sophicated or $2^2 = 4$ subpossibilities; hence total number of possibilities in this case is: $10 \cdot 4 = 40$;

or 4 of the 5 classical mathematical propositions (hence we have $C_5^4 = 5$ subpossibilities) – so 1 classical mathematical propositions remain either unchanged other neutro-sophicated or $2^1 = 2$ subpossibilities; hence total number of possibilities in this case is: $5 \cdot 2 = 10$;

or all 5 of the 5 classical mathematical propositions (hence we have $C_5^5 = 1$ subpossibility) – so no classical mathematical propositions remain.

Hence, the total number of Anti-*BE*-algebras is:

$$C_5^1 \cdot 2^{5-1} + C_5^2 \cdot 2^{5-2} + C_5^3 \cdot 2^{5-3} + C_5^4 \cdot 2^{5-4} + C_5^5 \cdot 2^{5-5} = 5 \cdot 16 + 10 \cdot 8 + 10 \cdot 4 + 5 \cdot 2 + 1 \cdot 1 = 211.$$

Theorem 3.11.

As a particular case, for *BE*-algebras, we have:

$$1 \text{ (classical) } BE\text{-algebra} + 31 \text{ Neutro-}BE\text{-algebras} + 211 \text{ Anti-}BE\text{-algebras} = 243 = 3^5 \text{ algebras.}$$

Where, $31 = 2^5 - 1$, and $211 = 3^5 - 2^5$.

Proof.

It results from the previous Theorem 3.10 and 3.11.

Theorem 3.12.

Let U be a nonempty finite or infinite universe of discourse, and S a nonempty finite or infinite subset of U . A classical Algebra is defined on S .

In general, for a given classical Algebra, having n operations (laws) and axioms altogether, for integer $n \geq 1$, there are 3^n total number of Algebra / NeutroAlgebras / AntiAlgebras as below:

1 (classical) Algebra, $(2^n - 1)$ Neutro-Algebras, and $(3^n - 2^n)$ Anti-Algebras.

The finite or infinite cardinal of set the classical algebra is defined upon, does not influence the numbers of Neutro-*BE*-algebras and Anti-*BE*-algebras.

Proof.

It is similar to Theorem 3.11, and based on Theorems 3.10 and 3.11.

Where 5 (total number of classical laws and axioms altogether) is extended/replaced by n .

5. Conclusion.

We have studied and presented the neutrosophic triplet (*BE*-algebra, Neutro-*BE*-algebra, Anti-*BE*-algebra) together with many examples, several properties and four theorems.

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Elementary Examination of NeutroAlgebras and AntiAlgebras viz-a-viz the Classical Number Systems

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Abstract

The objective of this paper is to examine NeutroAlgebras and AntiAlgebras viz-a-viz the classical number systems.

Keywords: NeutroAlgebra, AntiAlgebra, NeutroAlgebraic Structure, AntiAlgebraic Structure.

1 Introduction

The notions of NeutroAlgebra and AntiAlgebra were recently introduced by Florentin Smarandache.^[1] Smarandache in^[2] revisited the notions of NeutroAlgebra and AntiAlgebra and in^[3] he studied Partial Algebra, Universal Algebra, Effect Algebra and Boole's Partial Algebra and showed that NeutroAlgebra is a generalization of Partial Algebra. In the present Short Communication, we are going to examine NeutroAlgebras and AntiAlgebras viz-a-viz the classical number systems. For more details about NeutroAlgebras, AntiAlgebras, NeutroAlgebraic Structures and AntiAlgebraic Structures, the readers should see.^[1,3]

Let U be a universe of discourse and let X be a nonempty subset of U . Suppose that A is an item (concept, attribute, idea, proposition, theory, algebra, structure etc.) defined on the set X . By neutrosophication approach, X can be split into three regions namely: $\langle A \rangle$ the region formed by the sets of all elements where $\langle A \rangle$ is true with the degree of truth (T), $\langle antiA \rangle$ the region formed by the sets of all elements where $\langle A \rangle$ is false with the degree of falsity (F) and $\langle neutA \rangle$ the region formed by the sets of all elements where $\langle A \rangle$ is indeterminate (neither true nor false) with the degree of indeterminacy (I). It should be noted that depending on the application, $\langle A \rangle$, $\langle antiA \rangle$ and $\langle neutA \rangle$ may or may not be disjoint but they are exhaustive that is; their union is X . If A represents Function, Operation, Axiom, Algebra etc, then we can have the corresponding triplets $\langle Function, NeutroFunction, AntiFunction \rangle$, $\langle Operation, NeutroOperation, AntiOperation \rangle$, $\langle Axiom, NeutroAxiom, AntiAxiom \rangle$ and $\langle Algebra, NeutroAlgebra, AntiAlgebra \rangle$ etc.

Definition 1.1. [1]

- (i) A NeutroAlgebra X is an algebra which has at least one NeutroOperation or one NeutroAxiom that is; axiom that is true for some elements, indeterminate for other elements, and, false for other elements.
- (ii) An AntiAlgebra X is an algebra endowed with a law of composition such that the law is false for all the elements of X .

Definition 1.2. [1] Let X and Y be nonempty subsets of a universe of discourse U and let $f : X \rightarrow Y$ be a function. Let $x \in X$ be an element. We define the following with respect to $f(x)$ the image of x :

- (i) Inner-defined or Well-defined: This corresponds to $f(x) \in Y$ (True)(T). In this case, f is called a Total Inner-Function which corresponds to the Classical Function.
- (ii) Outer-defined: This corresponds to $f(x) \in U - Y$ (Falsehood) (F). In this case, f is called a Total Outer-Function or AntiFunction.
- (iii) Indeterminacy: This corresponds to $f(x) = \text{indeterminacy}$ (Indeterminate) (I); that is, the value $f(x)$ does exist, but we do not know it exactly. In this case, f is called a Total Indeterminate Function.

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2 Subject Matter

In what follows, we will consider the classical number systems $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$ of natural, integer, rational, real and complex numbers respectively and noting that $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C}$. Let $+, -, \times, \div$ be the usual binary operations of addition, subtraction, multiplication and division of numbers respectively. Using elementary approach, we will examine whether or not the abstract systems $(\mathbb{N}, *)$, $(\mathbb{Z}, *)$, $(\mathbb{Q}, *)$, $(\mathbb{R}, *)$, $(\mathbb{C}, *)$ are NeutroAlgebras or and AntiAlgebras where $* = +, -, \times, \div$.

(1) Let $X = \mathbb{N}$.

(i) It is clear that $(X, +)$ and (X, \times) are neither NeutroAlgebras nor AntiAlgebras.

(ii) For some $x, y \in X$, $x - y \in X$ (True) (Inner) or $x - y \notin X$ (False) (Outer). However, for all $x, y \in X$ with $x \leq y$, $x - y \notin X$ (False) (Outer) and for all $x, y \in X$ with $x > y$, we have $x - y \in X$ (True) (Inner). This shows that $-$ is a NeutroOperation over X and $\therefore (X, -)$ is a NeutroGroupoid. The operation $-$ is not commutative for all $x \in X$. This shows that $-$ is AntiCommutative over X . We claim that $-$ is NeuroAssociative over X .

Proof. For $x > y, z = 0$, we have $x - (y - z) = (x - y) - z$, or $x - y + 0 = x - y - 0 > 0$ (degree of Truth) (T). However, for $x > y, z \neq 0$, we have $x - (y - z) \neq (x - y) - z$ (degree of Falsehood) (F). For $x < y, c = 0$, we have $x - y + 0 = x - y - 0 < 0$ (degree of Indeterminacy) (I). This shows that $-$ is NeuroAssociative and $\therefore (X, -)$ is a NeutroSemigroup. \square

(iii) For all $x \in X$, $x \div 1 \in X$ (True) (Inner). For some $x, y \in X$, $x \div y \notin X$ (False) (Outer). However, if x is a multiple of y including 1, then $x \div y \in X$ (True) (Inner). This shows that \div is a NeutroOperation and therefore, (X, \div) is a NeutroGroupoid. It can be shown that \div is NeuroAssociative over X and therefore, (X, \div) is a NeutroSemigroup.

The equation $ax = b$ is not solvable for some $a, b \in X$. However, if b is a multiple of a including 1, then the equation is solvable and the solution is called a NeutroSolution. Also, the equation $acx^2 + bd = (ad + bc)x$ is not solvable for some $a, b, c, d \in X$. However, if b is a multiple of a including 1 and c is a multiple of d including 1, the equation is solvable and the solutions are called NeutroSolutions.

Let \circ be a binary operation defined for all $x, y \in X$ by

$$x \circ y = \begin{cases} 0 & \text{if } x = y \\ -\alpha & \text{if } x < y \\ -\beta & \text{if } x > y \end{cases}$$

where $\alpha, \beta \in \mathbb{N}$ such that $\alpha \leq \beta$. It is clear that \circ is an AntiOperation on X and $\therefore (X, \circ)$ is an AntiAlgebra.

(2) Let $X = \mathbb{Z}$.

(i) $(X, +)$ and (X, \times) are neither NeutroAlgebras nor AntiAlgebras.

(ii) For all $x, y, z \in X$ such that $x, y = 0, 1$, we have $x - y = y - x = 0 \in X$ (True), otherwise for other elements, the result is False (Outer) so that $-$ is NeuroCommutative over X . However, if $x, y, z = 0$, then $x - (y - z) = (x - y) - z = 0 \in X$ (True), otherwise for other elements, the result is False and consequently, $-$ is NeuroAssociative over X and hence $(X, -)$ is a NeutroSemigroup.

(iii) For all $x \in X$, $x \div \pm 1 \in X$ (True) (Inner). For all $x \in X$, $x \div 0 = \text{indeterminate}$ (Indeterminacy). For some $x, y \in X$, $x \div y \notin X$ (False) (Outer) however, if x is a multiple of y including ± 1 , then $x \div y \in X$ (True) (Inner). This shows that \div is a NeutroOperation over X and $\therefore (X, \div)$ is a NeutroGroupoid. It can also be shown that (X, \div) is a NeutroSemigroup.

The equation $ax = b$ is not solvable for some $a, b \in X$. If $a = 0$, the solution is indeterminate (Indeterminacy). However, if b is a multiple of a including ± 1 , then the equation is solvable and the solution is called a NeutroSolution. Also, the equation $acx^2 + (ad - bc)x - bd = 0$ is not solvable for some $a, b, c, d \in X$. However, if b is a multiple of a including ± 1 and c is a multiple of d including ± 1 , the equation is solvable and the solutions are called NeutroSolutions.

For all $x, y \in X$, let \circ be a binary operation defined by $x \circ y = \ln(xy)$. If $x, y = 0$, we have $x \circ y = \text{indeterminate}$ (Indeterminacy) (I). If $x > 0, y < 0$, we have $x \circ y = \text{indeterminate}$ (Indeterminacy) (I). If $x > 0, y > 0$, we have $x \circ y = \text{False}$ (F) except when $x = y = 1$. These show that \circ is a NeutroOperation over X and $\therefore (X, \circ)$ is a NeutroAlgebra.

Let \circ be a binary operation defined for all $x, y \in X$ by

$$x \circ y = \begin{cases} -1/2 & \text{if } x < y \\ 1/2 & \text{if } x > y \end{cases}$$

It is clear that \circ is an AntiOperation on X and $\therefore (X, \circ)$ is an AntiAlgebra.

(3) Let $X = \mathbb{Q}$.

- (i) $(X, +)$ and (X, \times) are neither NeutroAlgebras nor AntiAlgebras.
- (ii) For all $x, y, z \in X$ such that $x, y, z = 1$, we have $x - y = y - x = 0 \in X$ (True), otherwise for other elements, the result is False so that $-$ is NeuroCommutative over X . Also, if $x, y, z = 0$, then $x - (y - z) = (x - y) - z = 0 \in X$ (True), otherwise for other elements, the result is False and consequently, $-$ is NeuroAssociative over X and $(X, -)$ is a NeutroSemigroup.
- (iii) For all $0 \neq x, y \in X$, $x \div y \in X$ (True) (Inner) but for all $x \in X$, $x \div 0 = \text{indeterminate}$ (Indeterminacy). $\therefore (X, \div)$ is a NeutroAlgebra which we call a NeutroField.

For all $x, y \in X$, let \circ be a binary operation defined by $x \circ y = e^{x \div y}$. If $x, y = 0$, we have $x \circ y = \text{indeterminate}$ (Indeterminacy) (I). If $x > 0, y = 0$, we have $x \circ y = \text{indeterminate}$ (Indeterminacy) (I). If $x > 0, y > 0$, we have $x \circ y = \text{False}$ (F). These show that \circ is a NeutroOperation over X and $\therefore (X, \circ)$ is a NeutroAlgebra.

Let \circ be a binary operation defined for all $x, y \in X$ by

$$x \circ y = \begin{cases} -e & \text{if } x \leq y \\ e & \text{if } x \geq y \end{cases}$$

where e is the base of Napierian Logarithm. It is clear that \circ is an AntiOperation on X and $\therefore (X, \circ)$ is an AntiAlgebra.

(4) Let $X = \mathbb{R}$.

- (i) $(X, +)$ and (X, \times) are neither NeutroAlgebras nor PartialAlgebras.
- (ii) For all $x, y \in X$ such that $x, y = 0, \pm 1$, we have $x - y = y - x = 0 \in X$ (True), otherwise for other elements, the result is False so that $-$ is NeuroCommutative over X .
- (iii) For all $0 \neq x, y \in X$, $x \div y \in X$ (True) (Inner) but for all $x \in X$, $x \div 0 = \text{indeterminate}$ (Indeterminacy). It can be shown that \div is NeuroAssociative over X . Hence, (X, \div) is a NeutroSemigroup and therefore, it is a NeutroAlgebra which we call a NeutroField.

Let \circ be a binary operation defined for all $x, y \in X$ by

$$x \circ y = \begin{cases} -\sqrt{-1} & \text{if } x \leq y \\ \sqrt{-1} & \text{if } x \geq y \end{cases}$$

It is clear that \circ is an AntiOperation on X and $\therefore (X, \circ)$ is an AntiAlgebra.

(5) Let $X = \mathbb{C}$.

- (i) $(X, +)$ and (X, \times) are neither NeutroAlgebras nor AntiAlgebras.
- (ii) For all $z, w \in X$ such that $z, w = 0, \pm i$, we have $z - w = w - z = 0 \in X$ (True), otherwise for other elements, the result is False so that $-$ is NeuroCommutative over X .
- (iii) For all $0 \neq z, w \in X$, $z \div w \in X$ (True) (Inner) but for all $z \in X$, $z \div 0 = \text{indeterminate}$ (Indeterminacy). Therefore, (X, \div) is a NeutroAlgebra which we call a NeutroField.

Let \circ be a binary operation defined for all $z, w \in X$ by

$$z \circ w = \begin{cases} i & \text{if } |z| = |w| \\ j & \text{if } |z| \leq |w| \\ k & \text{if } |z| \geq |w| \end{cases}$$

where $ijk = -1$. It is clear that \circ is an AntiOperation on X and $\therefore (X, \circ)$ is an AntiAlgebra.

Theorem 2.1. For all prime number $n \geq 2$, $(\mathbb{Z}_n, +, \times)$ is a NeutroAlgebra called a NeutroField.

Proof. Suppose that $n \geq 2$ is a prime number. Clearly, 1 is the multiplicative identity element in \mathbb{Z}_n . For all $0 \neq x \in \mathbb{Z}_n$, there exist a unique $y \in \mathbb{Z}_n$ such that $x \times y = 1$ (True) (T). However, for $0 = x \in \mathbb{Z}_n$, there does not exist any unique $y \in \mathbb{Z}_n$ such that $x \times y = 1$ (False) (F). This shows that (\mathbb{Z}_n, \times) is a NeutroGroup. Since $(\mathbb{Z}_n, +)$ is an abelian group, it follows that $(\mathbb{Z}_n, +, \times)$ is a NeutroDivisionRing called a NeutroField. \square

3 Conclusion

We have in this paper examined NeutroAlgebras and AntiAlgebras viz-a-viz the classical number systems $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}$ of natural, integer, rational, real and complex numbers respectively. In our future papers, we hope to study more algebraic properties of NeutroAlgebras and NeutroSubalgebras and NeutroMorphisms between them.

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On Neutrosophic Quadruple Hypervector Spaces

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Abstract

The objective of this paper is to study Neutrosophic Quadruple Hypervector Spaces and present some of their basic definitions and properties. This paper generalizes the concept of Neutrosophic Hypervector spaces by presenting their Neutrosophic Quadruple forms. Some notions such as Neutrosophic hypersubspaces, linear combination, linearly dependence and linearly independence are generalized. Some interesting results and examples to illustrate the new concepts introduced are presented.

Keywords: Neutrosophic Quadruple (NQ), Neutrosophic Quadruple set, NQ Hypervector spaces, Super strong NQ Hypervector spaces, strong NQ Hypervector spaces, Weak NQ Hypervector spaces, NQ field, Neutrosophic field, NQ Hypersubspaces, NQ bases.

1 Introduction

Neutrosophy is a new branch of philosophy that studies the origin, nature, and scope of neutralities, as well as their interactions with different ideational spectra. Neutrosophic set and neutrosophic logic were introduced in 1995 by Smarandache as generalizations of fuzzy set and respectively intuitionistic fuzzy logic. In neutrosophic logic, each proposition has a degree of truth (T), a degree of indeterminacy (I), and a degree of falsity (F), where T, I, F are standard or non-standard subsets of $]^{-}0, 1^{+}[$, see [27,28,29].

The notion of neutrosophic algebraic structures was introduced by Kandasamy and Smarandache in 2006, see [32,33]. Since then, several researchers have studied the concepts and a great deal of literature has been produced. For example, Agboola and Akinleye introduced the concept of neutrosophic hypervector spaces in [1] and in [2] they studied neutrosophic vector spaces. In [34] Vasantha K., Ilanthenral K. and Smarandache F. introduced for the first time the concept of neutrosophic quadruple vector spaces over the classical fields \mathbb{R}, \mathbb{C} and \mathbb{Z}_p . A comprehensive review of neutrosophy, neutrosophic triplet set, neutrosophic quadruple set and neutrosophic algebraic structures can be found in [34,9,10,11,12,14,15,17,18,19,20,21,22,30,31].

The concept of hyperstructure was first introduced by Marty [16] in 1934 at the 8th congress of Scandinavian Mathematicians and then he established the definition of hypergroup in 1935 to analyze its properties and applied them to groups of rational algebraic functions. M. Krasner [13] introduced the notions of hyperring and hyperfield and use them as technical tools in the study of the approximation of valued fields. These concepts have been developed and generalized by many researchers.

The notion of hypervector spaces was introduced by M. Scafati Tallini [24] in 1988. Hypervector spaces have been further expanded by other researchers. For more detailed information on hypervector spaces, the reader should see [5,6,7,8,23,24,25,26].

The present paper is concerned with introducing the concept of neutrosophic quadruple hypervector spaces. Some of their elementary properties are presented.

2 Preliminaries

In this section, some basic definitions and properties that will be useful in this work are given.

Definition 2.1. A neutrosophic quadruple number is a number of the form (a, bT, cI, dF) where T, I, F have their usual neutrosophic logic meanings and $a, b, c, d \in \mathbb{R}$ or \mathbb{C} . The set NQ defined by $NQ = \{(a, bT, cI, dF) : a, b, c, d \in \mathbb{R} \text{ or } \mathbb{C}\}$ is called neutrosophic quadruple set.

Definition 2.2. ^[24] Suppose in an optimistic way we consider the prevalence order $T > I > F$. Then the combination of the usual Neutrosophic tools T, I, F are :

$$\begin{aligned} TI &= IT = \max\{T, I\} = T, \\ TF &= FT = \max\{T, F\} = T, \\ IF &= FI = \max\{I, F\} = I, \\ TT &= T^2 = T, \\ II &= I^2 = I, \\ FF &= F^2 = F \end{aligned}$$

Analogously, suppose in a pessimistic way we consider the prevalence order $T < I < F$. Then we have:

$$\begin{aligned} TI &= IT = \max\{T, I\} = I, \\ TF &= FT = \max\{T, F\} = F, \\ IF &= FI = \max\{I, F\} = F, \\ TT &= T^2 = T, \\ II &= I^2 = I, \\ FF &= F^2 = F \end{aligned}$$

We shall adopt the pessimistic way in this work.

The following operations are defined on NQ , for $x = (a, bT, cI, dF)$ and $y = (e, fT, gI, hF) \in NQ$ we have that

$$x + y = (a, bT, cI, dF) + (e, fT, gI, hF) = (a + e, (b + f)T, (c + g)I, (d + h)F) \text{ and}$$

$$x - y = (a, bT, cI, dF) - (e, fT, gI, hF) = (a - e, (b - f)T, (c - g)I, (d - h)F) \text{ are in } NQ.$$

For $x = (a, bT, cI, dF) \in NQ$ and $k \in \mathbb{R}$ where k is a scalar and x is a vector in NQ .

$$k.x = k.(a, bT, cI, dF) = (ka, kbT, kcI, kdF) \in NQ.$$

If $x = 0 = (0, 0, 0, 0) \in NQ$ usually termed as zero neutrosophic quadruple vector and for any scalar $k \in \mathbb{R}$ we have $k \cdot 0 = 0$.

Further

$$(k + p)x = kx + px, k(px) = (kp)x, k(x + y) = kx + ky.$$

for all $k, p \in \mathbb{R}$ and $x, y \in NQ$. $-x = (-a, -bT, -cI, -dF)$ which is in NQ .

Definition 2.3. Let $a = (a_1, a_2T, a_3I, a_4F), b = (b_1, b_2T, b_3I, b_4F) \in NQ$. Then

$$\begin{aligned} a \cdot b &= (a_1, a_2T, a_3I, a_4F) \cdot (b_1, b_2T, b_3I, b_4F) \\ &= (a_1b_1, (a_1b_2 + a_2b_1 + a_2b_2)T, (a_1b_3 + a_2b_3 + a_3b_1 + a_3b_2 + a_3b_3)I, \\ &\quad (a_1b_4 + a_2b_4 + a_3b_4 + a_4b_1 + a_4b_2 + a_4b_3 + a_4b_4)F). \end{aligned}$$

Theorem 2.4. ^[24] $(NQ, +)$ is an abelian group.

Theorem 2.5. ^[24] (NQ, \cdot) is a commutative monoid.

Theorem 2.6. ^[24] (NQ, \cdot) is not a group.

Theorem 2.7. ^[24] $(NQ, +, \cdot)$ is a commutative ring.

Theorem 2.8. ^[34] $(NQ, +) = \{(a, bT, cI, dF) | a, b, c, d \in \mathbb{R} \text{ or } \mathbb{C} \text{ or } \mathbb{Z}_p; p \text{ a prime}, +\}$ be the Neutrosophic quadruple group. Then $V = (NQ, +, o)$ is a Neutrosophic Quadruple vector space (NQ – vectorspace) over \mathbb{R} or \mathbb{C} or \mathbb{Z}_p , where 'o' is the special type of operation between V and \mathbb{R} (or \mathbb{C} or \mathbb{Z}_p) defined as scalar multiplication.

Definition 2.9. ^[34] Let $V = (NQ, +)$ be a NQ vector space over \mathbb{R} (or \mathbb{C} or \mathbb{Z}_p). A subset L of V is said to be NQ linearly dependent or simply dependent, if there exists distinct vectors $a_1, a_2, \dots, a_k \in L$ and scalars $d_1, d_2, \dots, d_k \in \mathbb{R}$ (or \mathbb{C} or \mathbb{Z}_p) not all zero such that $d_1 \circ a_1 + d_2 \circ a_2 + \dots + d_k \circ a_k = 0$.

We say the set of vectors a_1, a_2, \dots, a_k is NQ linearly independent if it is not NQ linearly dependent.

Definition 2.10. ^[34] Let $V = (NQ, +)$ be a NQ vector space over \mathbb{R} (or \mathbb{C} or \mathbb{Z}_p). A subset W of V is said to be Neutrosophic Quadruple vector subspace of V if W itself is a Neutrosophic Quadruple vector space over \mathbb{R} (or \mathbb{C} or \mathbb{Z}_p).

Definition 2.11. ^[1] Let $P(V)$ be the power set of a set V , $P^*(V) = P(V) - \{\emptyset\}$ and let K be a field. The quadruple $(V, +, \bullet, K)$ is called a hypervector space over a field K if:

1. $(V, +)$ is an abelian group.
2. $\bullet : K \times V \longrightarrow P^*(V)$ is a hyperoperation such that for all $k, m \in K$ and $u, v \in V$, the following conditions hold:
 - (a) $(k + m) \bullet u \subseteq (k \bullet u) + (m \bullet u)$,
 - (b) $k \bullet (u + v) \subseteq (k \bullet u) + (k \bullet v)$,
 - (c) $k \bullet (m \bullet u) = (km) \bullet u$, where $k \bullet (m \bullet u) = \{k \bullet v : v \in m \bullet u\}$,
 - (d) $(-k) \bullet u = k \bullet (-u)$,
 - (e) $u \in 1 \bullet u$.

A hypervector space is said to be strongly left distributive (resp. strongly right distributive) if equality holds in (a) (resp. in (b)). $(V, +, \bullet, K)$ is called a strongly distributive hypervector space if it is both strongly left and strongly right distributive.

Definition 2.12. ^[1] Let $(V, +, \bullet, K)$ be any strongly distributive hypervector space over a field K and let

$$V(I) = \langle V \cup (I) \rangle = \{u = (a, bI) : a, b \in V\}$$

be a set generated by V , and I . The quadruple $(V(I), +, \bullet, K)$ is called a weak neutrosophic strongly distributive hyper vector space over a field K .

For every element $u = (a, bI), v = (d, eI) \in V(I)$, and $k \in K$ we define

$$u + v = (a + d, (b + e)I) \in V(I),$$

$$k \bullet u = \{(x, yI) : x \in k \bullet a, y \in k \bullet b\}.$$

If K is a neutrosophic field, that is, $K = K(I)$, then the quadruple $(V(I), +, \bullet, K(I))$ is called a strong neutrosophic strongly distributive hyper vector space over a neutrosophic field $K(I)$.

For every element $u = (a, bI), v = (d, eI) \in V(I)$, and $\alpha = (k, mI) \in K(I)$, we define

$$u + v = (a, bI) + (d, eI) = (a + d, (b + e)I),$$

$$\alpha \bullet u = \{(x, yI) : (x \in k \bullet a, y \in k \bullet b \cup m \bullet a \cup m \bullet b)\}.$$

The elements of $V(I)$ are called neutrosophic vectors and the elements of $K(I)$ are called neutrosophic scalars. The zero neutrosophic vector of $V(I)$, $(0, 0I)$, is denoted by θ , the zero element $0 \in K$ is represented by $(0, 0I)$ in $K(I)$ and $1 \in K$ is represented by $(1, 0I) \in K(I)$.

Theorem 2.13. ^[2] Every strong neutrosophic hypervector space is a weak neutrosophic hypervector space

Theorem 2.14. ^[2] Every weak neutrosophic hypervector space is a strongly distributive hypervector space

3 Formulation of a Neutrosophic Quadruple(NQ) Hypervector Spaces and its Subspaces

In this section, we develop the concept of neutrosophic quadruple hypervector spaces and present some of their basic properties. Except otherwise stated, all neutrosophic quadruple numbers will be real neutrosophic quadruple numbers of the form (a, bT, cI, dF) where $a, b, c, d \in \mathbb{R}$. The elements of $V(T, I, F)$ will be called neutrosophic quadruple vectors and the elements of $K(I)$ and $K(T, I, F)$ will be called neutrosophic scalars and neutrosophic quadruple scalars respectively. $(0, 0T, 0I, 0F)$, the zero vector of $V(T, I, F)$ will be denoted by θ , the zero element of $K(T, I, F)$ will be denoted by $0 \in K$ while $1 \in K$ will be denoted by $(1, 0T, 0I, 0F)$ in $K(T, I, F)$.

Definition 3.1. Let $(V, +, \bullet, K)$ be any strongly distributive hypervector space over a field K and let

$$V(T, I, F) = \langle V \cup (T, I, F) \rangle = \{u = (a, bT, cI, dF) : a, b, c, d \in V\}.$$

be a set generated by V, T, I and F . The quadruple $(V(T, I, F), +, \bullet, K)$ is called a weak neutrosophic quadruple strongly distributive hypervector space over a field K .

For every element $u = (a, bT, cI, dF), v = (e, fT, gI, hF) \in V(T, I, F)$ and $k \in K$ we define

$$u + v = (a + e, (b + f)T, (c + g)I, (d + h)F) \in V(T, I, F),$$

$$k \bullet u = \{(r, xT, yI, zF) : r \in k \bullet a, x \in k \bullet b, y \in k \bullet c, z \in k \bullet d\}.$$

Definition 3.2. Let $(V, +, \bullet, K)$ be any strongly distributive hypervector space over a field K and let

$$V(T, I, F) = \langle V \cup (T, I, F) \rangle = \{u = (a, bT, cI, dF) : a, b, c, d \in V\}.$$

be a set generated by V, T, I and F . The quadruple $(V(T, I, F), +, \bullet, K(I))$ is called a strong neutrosophic quadruple strongly distributive hypervector space over a neutrosophic field $K(I)$.

For every element $u = (a, bT, cI, dF), v = (e, fT, gI, hF) \in V(T, I, F)$ and $\alpha = (k, mI) \in K(I)$, we define

$$u + v = (a + e, (b + f)T, (c + g)I, (d + h)F) \in V(T, I, F),$$

$$\alpha \bullet u = \{(r, xT, yI, zF) : (r \in k \bullet a, x \in k \bullet b, y \in k \bullet c \cup m \bullet a \cup m \bullet b \cup m \bullet c, z \in k \bullet d \cup m \bullet d)\}.$$

Definition 3.3. Let $(V, +, \bullet, K)$ be any strongly distributive hypervector space over a field K and let

$$V(T, I, F) = \langle V \cup (T, I, F) \rangle = \{u = (a, bT, cI, dF) : a, b, c, d \in V\}$$

be a set generated by V, T, I and F .

The quadruple $(V(T, I, F), +, \bullet, K(T, I, F))$ is called a super strong neutrosophic quadruple strongly distributive hypervector space over a neutrosophic field $K(T, I, F)$.

For every element $u = (a, bT, cI, dF), v = (e, fT, gI, hF) \in V(T, I, F)$ and $\alpha = (k, mT, nI, tF) \in K(T, I, F)$, we define

$$u + v = (a + e, (b + f)T, (c + g)I, (d + h)F) \in V(T, I, F),$$

$$\alpha \bullet u = \{(r, xT, yI, zF) : r \in k \bullet a, x \in k \bullet b \cup m \bullet a \cup m \bullet b, y \in k \bullet c \cup m \bullet c \cup n \bullet a \cup n \bullet b \cup n \bullet c, z \in k \bullet d \cup m \bullet d \cup n \bullet d \cup t \bullet a \cup t \bullet b \cup t \bullet c \cup t \bullet d\}.$$

Example 3.4. Let n be a positive integer and let $V(T, I, F) = \mathbb{R}^n(T, I, F)$ denote the neutrosophic quadruple set of column neutrosophic quadruple vectors of length n with entries from the field \mathbb{R} :

$$\mathbb{R}^n(T, I, F) = \left\{ \begin{pmatrix} (a_1, b_1T, c_1I, d_1F) \\ (a_2, b_2T, c_2I, d_2F) \\ \vdots \\ (a_n, b_nT, c_nI, d_nF) \end{pmatrix} : a_i, b_i, c_i, d_i \in \mathbb{R}, \quad i = 1, 2, \dots, n \right\}$$

For all

$$u = \begin{pmatrix} (a_1, b_1T, c_1I, d_1F) \\ (a_2, b_2T, c_2I, d_2F) \\ \vdots \\ (a_n, b_nT, c_nI, d_nF) \end{pmatrix}, v = \begin{pmatrix} (e_1, f_1T, g_1I, h_1F) \\ (e_2, f_2T, g_2I, h_2F) \\ \vdots \\ (e_n, f_nT, g_nI, h_nF) \end{pmatrix} \in V(T, I, F)$$

and $k \in K$ define:

$$u + v = \begin{pmatrix} (a_1, b_1T, c_1I, d_1F) \\ (a_2, b_2T, c_2I, d_2F) \\ \vdots \\ (a_n, b_nT, c_nI, d_nF) \end{pmatrix} + \begin{pmatrix} (e_1, f_1T, g_1I, h_1F) \\ (e_2, f_2T, g_2I, h_2F) \\ \vdots \\ (e_n, f_nT, g_nI, h_nF) \end{pmatrix}$$

$$= \begin{pmatrix} (a_1 + e_1, (b_1 + f_1)T, (c_1 + g_1)I, (d_1 + h_1)F) \\ (a_2 + e_2, (b_2 + f_2)T, (c_2 + g_2)I, (d_2 + h_2)F) \\ \vdots \\ (a_n + e_n, (b_n + f_n)T, (c_n + g_n)I, (d_n + h_n)F) \end{pmatrix}$$

and

$$k \bullet \begin{pmatrix} (a_1, b_1T, c_1I, d_1F) \\ (a_2, b_2T, c_2I, d_2F) \\ \vdots \\ (a_n, b_nT, c_nI, d_nF) \end{pmatrix} = \left\{ \begin{pmatrix} (r_1, x_1T, y_1I, z_1F) \\ (r_2, x_2T, y_2I, z_2F) \\ \vdots \\ (r_n, x_nT, y_nI, z_nF) \end{pmatrix} : \begin{matrix} r_1 \in k \bullet a_1, x_1 \in k \bullet b_1, y_1 \in k \bullet c_1, z_1 \in k \bullet d_1 \\ r_2 \in k \bullet a_2, x_2 \in k \bullet b_2, y_2 \in k \bullet c_2, z_2 \in k \bullet d_2 \\ \vdots \\ r_n \in k \bullet a_n, x_n \in k \bullet b_n, y_n \in k \bullet c_n, z_n \in k \bullet d_n \end{matrix} \right\}.$$

Then $(V(T, I, F), +, \bullet, K)$ is a weak neutrosophic quadruple strongly distributive hypervector space over the field K .

Example 3.5. Let $V(T, I, F) = R^2(T, I, F)$ and let $K = R(I)$. For all $u = ((a_1, b_1T, c_1I, d_1F), (e_1, f_1T, g_1I, h_1F))$, $v = ((a_2, b_2T, c_2I, d_2F), (e_2, f_2T, g_2I, h_2F)) \in V(T, I, F)$ and $\alpha = (k, mI) \in K(I)$, define:

$$u + v = ((a_1 + a_2, (b_1 + b_2)T, (c_1 + c_2)I, (d_1 + d_2)F), (e_1 + e_2, (f_1 + f_2)T, (g_1 + g_2)I, (h_1 + h_2)F)).$$

$$\begin{aligned} \alpha \bullet u &= \{((r, xT, yI, zF), (p, qT, sI, tF)) : \\ &\quad (r \in k \bullet a_1, x \in k \bullet b_1, y \in k \bullet c_1 \cup m \bullet a_1 \cup m \bullet b_1 \cup m \bullet c_1, z \in k \bullet d_1 \cup m \bullet d_1) \\ &\quad (p \in k \bullet e_1, q \in k \bullet f_1, s \in k \bullet g_1 \cup m \bullet e_1 \cup m \bullet f_1 \cup m \bullet g_1, t \in k \bullet h_1 \cup m \bullet h_1)\}. \end{aligned}$$

Then $(V(T, I, F), +, \bullet, K(I))$ is an strong neutrosophic quadruple strongly distributive hypervector space over the neutrosophic field $K(I)$.

Example 3.6. Let $V(T, I, F) = R^2(T, I, F)$ and let $K = R(T, I, F)$. For all $u = ((a_1, b_1T, c_1I, d_1F), (e_1, f_1T, g_1I, h_1F))$, $v = ((a_2, b_2T, c_2I, d_2F), (e_2, f_2T, g_2I, h_2F)) \in V(T, I, F)$ and $\alpha = (k, mT, nI, wF) \in K(T, I, F)$, define:

$$u + v = ((a_1 + a_2, (b_1 + b_2)T, (c_1 + c_2)I, (d_1 + d_2)F), (e_1 + e_2, (f_1 + f_2)T, (g_1 + g_2)I, (h_1 + h_2)F)).$$

$$\begin{aligned} \alpha \bullet u &= \{((r, xT, yI, zF), (p, qT, sI, tF)) : \\ &\quad (r \in k \bullet a_1, x \in k \bullet b_1 \cup m \bullet a_1 \cup m \bullet b_1, y \in k \bullet c_1 \cup m \bullet c_1 \cup n \bullet a_1 \cup n \bullet b_1 \cup n \bullet c_1, \\ &\quad z \in k \bullet d_1 \cup m \bullet d_1 \cup n \bullet d_1 \cup w \bullet a_1 \cup w \bullet b_1 \cup w \bullet c_1 \cup w \bullet d_1) \\ &\quad (p \in k \bullet e_1, q \in k \bullet f_1 \cup m \bullet e_1 \cup m \bullet f_1, s \in k \bullet g_1 \cup m \bullet g_1 \cup n \bullet e_1 \cup n \bullet f_1 \cup n \bullet g_1, \\ &\quad t \in k \bullet h_1 \cup m \bullet h_1 \cup n \bullet h_1 \cup w \bullet e_1 \cup w \bullet f_1 \cup w \bullet g_1 \cup w \bullet h_1)\}. \end{aligned}$$

Then $(V(T, I, F), +, \bullet, K(T, I, F))$ is a super strong neutrosophic quadruple strongly distributive hypervector space over the neutrosophic quadruple field $K(T, I, F)$.

From here on, every weak(strong [super strong]) neutrosophic quadruple strongly distributive hypervector space will simply be called a weak(resp.(strong [super strong])) NQ-Hypervector space.

Proposition 3.7. .

1. Every super strong NQ-Hypervector space is a strong NQ-Hypervector space.
2. Every super strong NQ-Hypervector space is a weak NQ-Hypervector space.
3. Every strong NQ-Hypervector space is a weak NQ-Hypervector space.

Proof:

1. This is true, since $K(I) \subseteq K(T, I, F)$.
2. This is true, since $K \subseteq K(T, I, F)$.
3. This is true, since $K \subseteq K(I)$.

Proposition 3.8. Every weak NQ-Hypervector space is a strongly distributive hypervector space.

Proof: Suppose that $V(T, I, F)$ is a weak NQ-Hypervector space over a field K . That $(NQ, +)$ is a vector space is seen in [4].

Let $u = (a, bT, cI, dF)$, $v = (e, fT, gI, hF) \in V(T, I, F)$ and $k, m \in K$ be arbitrary. Then

$$\begin{aligned}
(1). \quad k \bullet u + m \bullet u &= \{(p, qT, rI, sF) : p \in k \bullet a, q \in k \bullet b, r \in k \bullet c, s \in k \bullet d\} \\
&\quad + \{(t, wT, xI, yF) : t \in m \bullet a, w \in m \bullet b, x \in m \bullet c, y \in m \bullet d\} \\
&= \{(p+t, (q+w)T, (r+x)I, (s+y)F) : p+t \in k \bullet a + m \bullet a, q+w \in k \bullet b + m \bullet b, \\
&\quad r+x \in k \bullet c + m \bullet c, s+y \in k \bullet d + m \bullet d\}.
\end{aligned}$$

Also

$$\begin{aligned}
(k+m) \bullet u &= \{(p', q'T, r'I, s'F) : p' \in (k+m) \bullet a, q' \in (k+m) \bullet b, r' \in (k+m) \bullet c, s' \in (k+m) \bullet d\} \\
&= \{(p', q'T, r'I, s'F) : p' \in k \bullet a + m \bullet a, q' \in k \bullet b + m \bullet b, r' \in k \bullet c + m \bullet c, s' \in k \bullet d + m \bullet d\} \\
&= k \bullet u + m \bullet u. \\
(2). \quad k \bullet u + k \bullet v &= \{(p, qT, rI, sF) : p \in k \bullet a, q \in k \bullet b, r \in k \bullet c, s \in k \bullet d\} \\
&\quad + \{(t, wT, xI, yF) : t \in k \bullet e, w \in k \bullet f, x \in k \bullet g, y \in k \bullet h\} \\
&= \{(p+t, (q+w)T, (r+x)I, (s+y)F) : p+t \in k \bullet a + k \bullet e, q+w \in k \bullet b + k \bullet f, \\
&\quad r+x \in k \bullet c + k \bullet g, s+y \in k \bullet d + k \bullet h\}.
\end{aligned}$$

Also,

$$\begin{aligned}
k \bullet (u+v) &= k \bullet (a+e, (b+f)T, (c+g)I, (d+h)F) \\
&= \{(p', q'T, r'I, s'F) : p' \in k \bullet (a+e), q' \in k \bullet (b+f), r' \in k \bullet (c+g), s' \in k \bullet (d+h)\} \\
&= \{(p', q'T, r'I, s'F) : p' \in k \bullet a + k \bullet e, q' \in k \bullet b + k \bullet f, r' \in k \bullet c + k \bullet g, \\
&\quad s' \in k \bullet d + k \bullet h\} \\
&= k \bullet u + k \bullet v.
\end{aligned}$$

$$\begin{aligned}
(3). \quad k \bullet (m \bullet u) &= k \bullet \{(p, qT, rI, sF) : p \in m \bullet a, q \in m \bullet b, r \in m \bullet c, s \in m \bullet d\} \\
&= \{(p', q'T, r'I, s'F) : p' \in k \bullet p, q' \in k \bullet q, r' \in k \bullet r, s' \in k \bullet s\} \\
&= \{(p', q'T, r'I, s'F) : p' \in k \bullet (m \bullet a), q' \in k \bullet (m \bullet b), r' \in (m \bullet c), s' \in (m \bullet d)\} \\
&= \{(p', q'T, r'I, s'F) : p' \in (km) \bullet a, q' \in (km) \bullet b, r' \in (km) \bullet c, s' \in (km) \bullet d\} \\
&= (km) \bullet (a, bT, cI, dF) \\
&= (km) \bullet u.
\end{aligned}$$

$$\begin{aligned}
(4). \quad (-k) \bullet u &= \{(p, qT, rI, sF) : p \in (-k) \bullet a, q \in (-k) \bullet b, r \in (-k) \bullet c, s \in (-k) \bullet d\} \\
&= \{(p, qT, rI, sF) : p \in k \bullet (-a), q \in k \bullet (-b), r \in k \bullet (-c), s \in k \bullet (-d)\} \\
&= k \bullet (-a, -bT, -cI, -dF) \\
&= k \bullet (-u).
\end{aligned}$$

$$\begin{aligned}
(5). \quad 1 \bullet u &= \{(p, qT, rI, sF) : p \in 1 \bullet a, q \in 1 \bullet b, r \in 1 \bullet c, s \in 1 \bullet d\} \\
&= \{(a, bT, cI, dF) : a \in 1 \bullet a, b \in 1 \bullet b, c \in 1 \bullet c, d \in 1 \bullet d\}.
\end{aligned}$$

Showing that $u \in 1 \bullet u$.

Therefore we say that $V(T, I, F)$ is a strongly distributive hypervector space.

Proposition 3.9. Let $V(T, I, F)$ be a super strong (strong) NQ-Hypervector space over a neutrosophic quadruple field $K(T, I, F)$ (neutrosophic field $K(I)$). Then

1. $V(T, I, F)$ generally is not a strongly distributive hypervector space.
2. $V(T, I, F)$ always contain a strongly distributive hypervector space

Proposition 3.10. Let $(V(T, I, F), +_1, \bullet_1)$ and $(U(T, I, F), +_2, \bullet_2)$ be any two super strong NQ-Hypervector space over a neutrosophic quadruple field $K(T, I, F)$. Let

$$\begin{aligned}
V(T, I, F) \times U(T, I, F) &= \{((v, v_1T, v_2I, v_3F), (u, u_1T, u_2I, u_3F)) : \\
&\quad (v, v_1T, v_2I, v_3F) \in V(T, I, F), (u, u_1T, u_2I, u_3F) \in U(T, I, F)\}.
\end{aligned}$$

For all

$$x = ((v, v_1T, v_2I, v_3F), (u, u_1T, u_2I, u_3F)), y = ((v', v'_1T, v'_2I, v'_3F), (u', u'_1T, u'_2I, u'_3F)) \in V(T, I, F) \times U(T, I, F)$$

$$\text{and } \alpha = (k, k_1T, k_2I, k_3F) \in K(T, I, F)$$

$$x + y = ((v + v', (v_1 + v'_1)T, (v_2 + v'_2)I, (v_3 + v'_3)F), (u + u', (u_1 + u'_1)T, (u_2 + u'_2)I, (u_3 + u'_3)F)).$$

$$\begin{aligned}
\alpha \bullet x &= \{((p, p_1T, p_2I, p_3F), (q, q_1T, q_2I, q_3F)) : \\
&\quad (p \in k \bullet v, p_1 \in k \bullet v_1 \cup k_1 \bullet v \cup k_1 \bullet v_1, p_2 \in k \bullet v_2 \cup k_1 \bullet v_2 \cup k_2 \bullet v \cup k_2 \bullet v_1 \cup k_2 \bullet v_2, \\
&\quad p_3 \in k \bullet v_3 \cup k_1 \bullet v_3 \cup k_2 \bullet v_3 \cup k_3 \bullet v \cup k_3 \bullet v_1 \cup k_3 \bullet v_2 \cup k_3 \bullet v_3) \\
&\quad (q \in k \bullet u, q_1 \in k \bullet u_1 \cup k_1 \bullet u \cup k_1 \bullet u_1, q_2 \in k \bullet u_2 \cup k_1 \bullet u_2 \cup k_2 \bullet u \cup k_2 \bullet u_1 \cup k_2 \bullet u_2, \\
&\quad q_3 \in k \bullet u_3 \cup k_1 \bullet u_3 \cup k_2 \bullet u_3 \cup k_3 \bullet u \cup k_3 \bullet u_1 \cup k_3 \bullet u_2 \cup k_3 \bullet u_3)\}.
\end{aligned}$$

Then $(V(T, I, F) \times U(T, I, F), +, \bullet, K(T, I, F))$ is a super strong NQ-Hypervector space.

Proposition 3.11. Let $(V(T, I, F), +_1, \bullet_1)$ and $(U(T, I, F), +_2, \bullet_2)$ be any two strong NQ-Hypervector space over a neutrosophic field $K(I)$. Let

$$V(T, I, F) \times U(T, I, F) = \{((v, v_1T, v_2I, v_3F), (u, u_1T, u_2I, u_3F)) : (v, v_1T, v_2I, v_3F) \in V(T, I, F), (u, u_1T, u_2I, u_3F) \in U(T, I, F)\}$$

for all

$$x = ((v, v_1T, v_2I, v_3F), (u, u_1T, u_2I, u_3F)), y = ((v', v'_1T, v'_2I, v'_3F), (u', u'_1T, u'_2I, u'_3F)) \in V(T, I, F) \times U(T, I, F) \text{ and } \alpha = (k, k_1I) \in K(I)$$

$$x + y = ((v + v', (v_1 + v'_1)T, (v_2 + v'_2)I, (v_3 + v'_3)F), (u + u', (u_1 + u'_1)T, (u_2 + u'_2)I, (u_3 + u'_3)F)).$$

$$\alpha \bullet x = \{((p, p_1T, p_2I, p_3F), (q, q_1T, q_2I, q_3F)) : (p \in k \bullet v, p_1 \in k \bullet v_1, p_2 \in k \bullet v_2 \cup k_1 \bullet v \cup k_1 \bullet v_1 \cup k_1 \bullet v_2, p_3 \in k \bullet v_3 \cup k_1 \bullet v_3) (q \in k \bullet u, q_1 \in k \bullet u_1, q_2 \in k \bullet u_2 \cup k_1 \bullet u \cup k_1 \bullet u_1 \cup k_1 \bullet u_2, q_3 \in k \bullet u_3 \cup k_1 \bullet u_3)\}.$$

Then $(V(T, I, F) \times U(T, I, F), +, \bullet, K(I))$ is a strong NQ-Hypervector space.

Proposition 3.12. Let $(V(T, I, F), +_1, \bullet_1)$ and $(U(T, I, F), +_2, \bullet_2)$ be any two weak NQ-Hypervector spaces over a field K . Let

$$V(T, I, F) \times U(T, I, F) = \{((v, v_1T, v_2I, v_3F), (u, u_1T, u_2I, u_3F)) : (v, v_1T, v_2I, v_3F) \in V(T, I, F), (u, u_1T, u_2I, u_3F) \in U(T, I, F)\}.$$

For all

$$x = ((v, v_1T, v_2I, v_3F), (u, u_1T, u_2I, u_3F)), y = ((v', v'_1T, v'_2I, v'_3F), (u', u'_1T, u'_2I, u'_3F)) \in V(T, I, F) \times U(T, I, F) \text{ and } k \in K$$

$$x + y = ((v + v', (v_1 + v'_1)T, (v_2 + v'_2)I, (v_3 + v'_3)F), (u + u', (u_1 + u'_1)T, (u_2 + u'_2)I, (u_3 + u'_3)F)).$$

$$k \bullet x = \{((p, p_1T, p_2I, p_3F), (q, q_1T, q_2I, q_3F)) : (p \in k \bullet v, p_1 \in k \bullet v_1, p_2 \in k \bullet v_2, p_3 \in k \bullet v_3) (q \in k \bullet u, q_1 \in k \bullet u_1, q_2 \in k \bullet u_2, q_3 \in k \bullet u_3)\}.$$

Then $(V(T, I, F) \times U(T, I, F), +, \bullet, K)$ is a weak NQ-Hypervector space.

Proposition 3.13. Let $V(T, I, F)$ be any super strong NQ-Hypervector space over a neutrosophic quadruple field $K(T, I, F)$, let $U(T, I, F)$ be any strong NQ-Hypervector space over a neutrosophic field $K(I)$ and let $W(T, I, F)$ be any weak NQ-Hypervector space over a field K . Then

1. $(V(T, I, F) \times U(T, I, F), +, \bullet, K(I))$ is a strong NQ-Hypervector space.
2. $(V(T, I, F) \times W(T, I, F), +, \bullet, K)$ is a weak NQ-Hypervector space.
3. $(U(T, I, F) \times W(T, I, F), +, \bullet, K)$ is a weak NQ-Hypervector space.

Proof:

1. From 1 of [3.7], we know that every super strong NQ-Hypervector space is a strong NQ-Hypervector space. Then by applying [3.11] to this, we obtained the required result.
2. From 2 of [3.7], we know that every super strong NQ-Hypervector space is a weak NQ-Hypervector space. Then by [3.12] the proof follows.
3. From 3 of [3.7], we know that every strong NQ-Hypervector space is a weak NQ-Hypervector space. Then by [3.12] the proof follows.

Definition 3.14. A nonempty subset $N(T, I, F)$ of a super strong NQ-Hypervector space $(V(T, I, F), +, \bullet, K(T, I, F))$ over a neutrosophic quadruple field $K(T, I, F)$ is called a super strong NQ-Hypersubspace of $V(T, I, F)$ if $(N(T, I, F), +, \bullet, K(T, I, F))$ is itself a super strong NQ-Hypervector space over $K(T, I, F)$. It is essential that $N(T, I, F)$ contains a proper subset which is a Hypervector space over K .

Definition 3.15. A nonempty subset $N(T, I, F)$ of a strong NQ-Hypervector space $(V(T, I, F), +, \bullet, K(I))$ over a neutrosophic field $K(I)$ is called a strong NQ-Hypersubspace of $V(T, I, F)$ if $(N(T, I, F), +, \bullet, K(I))$ is itself a strong NQ-Hypervector space over $K(I)$. It is essential that $N(T, I, F)$ contains a proper subset which is a Hypervector space over K .

Proposition 3.16. Let $N[T, I, F]$ be a subset of a super strong NQ-Hypervector space $(V(T, I, F), +, \bullet, K(T, I, F))$ over a neutrosophic quadruple field $K(T, I, F)$. Then $N(I, T, F)$ is a super strong NQ-Hypersubspace of $V(T, I, F)$ if and only if for all $u = (a, bT, cI, dF), v = (e, fT, gI, hF) \in V(T, I, F)$ and $\alpha = (k, mT, nI, tF) \in K(T, I, F)$ the following conditions hold:

1. $N[T, I, F] \neq \emptyset$,
2. $u + v \in N[T, I, F]$,
3. $\alpha \bullet v \subseteq N[T, I, F]$,
4. $N[T, I, F]$ contains a proper subset which is a hypervector space over K .

Proof:

If $N(T, I, F)$ is a super strong NQ-Hypersubspace of $V(T, I, F)$, then obviously conditions 1, 2, 3 and 4 hold.

Conversely, let $N[T, I, F]$ be a subset of $V(T, I, F)$ such that $N(T, I, F)$ satisfies the four conditions 1, 2, 3 and 4.

To proof that $N(T, I, F)$ is a NQ-Hypersubspace of $V(T, I, F)$. It is enough to prove that

1. $N(T, I, F)$ has a zero NQ-vector.
2. Each NQ-vector in $N(T, I, F)$ has an additive inverse.

Since $N(T, I, F)$ is non-empty, let $u = (a, bT, cI, dF) \in N(T, I, F)$.

Now for $(0, 0T, 0I, 0F) \in K(T, I, F)$ and by condition 3 we have that

$$(0, 0T, 0I, 0F) \bullet u = (0, 0T, 0I, 0F) \bullet (a, bT, cI, dF) \subseteq N(T, I, F) \implies \theta \in N(T, I, F).$$

Therefore $N(T, I, F)$ has a zero vector. Again, since $-(1, 0T, 0I, 0F) \in K(T, I, F)$ then

$$-(1, 0T, 0I, 0F) \bullet u = -(1, 0T, 0I, 0F) \bullet (a, bT, cI, dF) \subseteq N \implies -u \in N(T, I, F).$$

Hence each NQ-vector in $N(T, I, F)$ has an additive inverse.

Proposition 3.17. Let $N[T, I, F]$ be a subset of a strong NQ-Hypervector space $(V(T, I, F), +, \bullet, K(I))$ over a neutrosophic field $K(I)$. Then $N(I, T, F)$ is a strong NQ-hypersubspace of $V(T, I, F)$ if and only if for all $u = (a, bT, cI, dF), v = (e, fT, gI, hF) \in V(T, I, F)$ and $\alpha = (k, mI) \in K(I)$ the following conditions hold:

1. $N[T, I, F] \neq \emptyset$,
2. $u + v \in N[T, I, F]$
3. $\alpha \bullet v \subseteq N[T, I, F]$
4. $N[T, I, F]$ contains a proper subset which is a hypervector space over K

Proof : Follow similar approach as the proof of [3.16](#) above.

Corollary 3.18. Let $N[T, I, F]$ be a NQ-hypersubspace of a NQ-hypervector space $V(T, I, F)$ if and only if

1. $N[T, I, F]$ is non-empty.
2. $\alpha \bullet u + \beta \bullet v \subseteq N[T, I, F]$, for all $\alpha = (k_1, m_1T, n_1I, r_1F), \beta = (k_2, m_2T, n_2I, r_2F) \in K(T, I, F)$ and $u = (a, bT, cI, dF), v = (e, fT, gI, hF) \in N[T, I, F]$.
3. $N[T, I, F]$ contains a proper subset which is a hypervector space over K .

Example 3.19. Let $V(T, I, F)$ be a super strong NQ-Hypervector space defined in Example [3.6](#).

Let $N(T, I, F) = K(T, I, F) \times \{(0, 0T, 0I, 0F)\} \subseteq V(T, I, F)$

Then $N(T, I, F)$ is a super strong NQ-Hypersubspace

Proof: Since $\theta = ((0, 0T, 0I, 0F), (0, 0T, 0I, 0F)) \in N(T, I, F)$. Then $N(T, I, F) \neq \emptyset$

Now let

$u = ((a_1, b_1T, c_1I, d_1F), (0, 0T, 0I, 0F)), v = ((a_2, b_2T, c_2I, d_2F), (0, 0T, 0I, 0F)) \in N(T, I, F)$ and $\alpha = (k, mT, nI, wF), \beta = (k', m'T, n'I, w'F) \in K(T, I, F)$, with $a_1, b_1, c_1, d_1, a_2, b_2, c_2, d_2 \in N$ and $k, m, n, w, k', m', n', w' \in K$

Then $\alpha \bullet u + \beta \bullet v$

$$\begin{aligned}
 &= (k, mT, nI, wF) \bullet [(a_1, b_1T, c_1I, d_1F), (0, 0T, 0I, 0F)] + (k', m'T, n'I, w'F) \bullet [(a_2, b_2T, c_2I, d_2F), (0, 0T, 0I, 0F)] \\
 &\subseteq \{((x, yT, zI, tF), (p, qT, rI, sF)) : x \in k \bullet a_1, y \in k \bullet b_1 \cup m \bullet a_1 \cup m \bullet b_1, z \in k \bullet c_1 \cup m \bullet c_1 \cup n \bullet a_1 \cup n \bullet b_1, t \in k \bullet d_1 \cup m \bullet d_1 \cup n \bullet d_1 \cup w \bullet a_1 \cup w \bullet b_1 \cup w \bullet c_1 \cup w \bullet d_1, p \in k \bullet 0, q \in k \bullet 0 \cup m \bullet 0 \cup m \bullet 0, r \in k \bullet 0 \cup m \bullet 0 \cup n \bullet 0 \cup n \bullet 0, s \in k \bullet 0 \cup m \bullet 0 \cup n \bullet 0 \cup w \bullet 0 \cup w \bullet 0 \cup w \bullet 0\} \\
 &+ \{((x', y'T, z'I, t'F), (p', q'T, r'I, s'F)) : x' \in k' \bullet a_2, y' \in k' \bullet b_2 \cup m' \bullet a_2 \cup m' \bullet b_2, z' \in k' \bullet c_2 \cup m' \bullet c_2 \cup n' \bullet a_2 \cup n' \bullet b_2, t' \in k' \bullet d_2 \cup m' \bullet d_2 \cup n' \bullet d_2 \cup w' \bullet a_2 \cup w' \bullet b_2 \cup w' \bullet c_2 \cup w' \bullet d_2, p' \in k' \bullet 0, q' \in k' \bullet 0 \cup m' \bullet 0 \cup m' \bullet 0, r' \in k' \bullet 0 \cup m' \bullet 0 \cup n' \bullet 0 \cup n' \bullet 0, s' \in k' \bullet 0 \cup m' \bullet 0 \cup n' \bullet 0 \cup w' \bullet 0 \cup w' \bullet 0 \cup w' \bullet 0\} \\
 &= \{((x_1, y_1T, z_1I, t_1F), (x'_1, y'_1T, z'_1I, t'_1F)) : x_1 \in k \bullet a_1 + k' \bullet a_2, \\
 &y_1 \in k \bullet b_1 + k' \bullet b_2 \cup m \bullet a_1 + m' \bullet a_2 \cup m \bullet b_1 + m' \bullet b_2, z_1 \in k \bullet c_1 + k' \bullet c_2 \cup m \bullet c_1 + m' \bullet c_2 \cup n \bullet a_1 + n' \bullet a_2 \cup n \bullet b_1 + n' \bullet b_2 \cup n \bullet c_1 + n' \bullet c_2, t \in k \bullet d_1 + k' \bullet d_2 \cup m \bullet d_1 + m' \bullet d_2 \cup n \bullet d_1 + n' \bullet d_2 \cup w \bullet a_1 + w' \bullet a_2 \cup w \bullet b_1 + w' \bullet b_2 \cup w \bullet c_1 + w' \bullet c_2 \cup w \bullet d_1 + w' \bullet d_2, x'_1 \in 0, y'_1 \in 0, z'_1 \in 0, t'_1 \in 0\} \subseteq N(T, I, F). \\
 &\Rightarrow \alpha \bullet u + \beta \bullet v \subseteq N(T, I, F).
 \end{aligned}$$

Lastly, we can see from the definition of $N(T, I, F)$ that $N(T, I, F)$ contains a proper subset which is a hypervector space over K .

To this end we can conclude that $N(T, I, F)$ is a super strong NQ-Hypervector space.

Proposition 3.20. *The intersection of any two*

1. *super strong NQ-Hypersubspaces of a super strong NQ-Hypervector space $V(T, I, F)$ over a neutrosophic quadruple field (K, I, F) is again a super strong NQ-Hypersubspace of $V(T, I, F)$.*
2. *strong NQ-Hypersubspaces of a strong NQ-Hypervector space $V(T, I, F)$ over a neutrosophic field $K(I)$ is again a strong NQ-Hypersubspace of $V(T, I, F)$.*
3. *weak NQ-Hypersubspaces of a weak NQ-Hypervector space $V(T, I, F)$ over a field K is again a weak NQ-Hypersubspace of $V(T, I, F)$.*

Proof: Same as in classical case.

Proposition 3.21. *Let $S(T, I, F)$ be a super strong NQ-Hypersubspace, $U(T, I, F)$ be a strong NQ-Hypersubspace and $W(T, I, F)$ be weak NQ-Hypersubspace of a super strong NQ-Hypervector space $(V(T, I, F), +, \bullet, K(T, I, F))$, strong NQ-Hypervector space $(V(T, I, F), +, \bullet, K(I))$ and weak NQ-Hypervector space $(V(T, I, F), +, \bullet, K)$ respectively. Then*

1. *$S(T, I, F) \cap U(T, I, F)$ is a strong NQ-Hypersubspace of strong NQ-Hypersubspace $(V(T, I, F), +, \bullet, K(I))$.*
2. *$S(T, I, F) \cap W(T, I, F)$ is a weak NQ-Hypersubspace of weak NQ-Hypersubspace $(V(T, I, F), +, \bullet, K)$.*
3. *$U(T, I, F) \cap W(T, I, F)$ is a weak NQ-Hypersubspace of weak NQ-Hypersubspace $(V(T, I, F), +, \bullet, K)$.*

Proof:

1. By 1 of [3.7] we have that every super strong NQ-Hypervector space is a strong NQ-Hypervector space. Then by 2 of [3.20] the proof follows.
2. By applying 2 of [3.7] and 3 of [3.20] the proof follows easily.
3. By 3 of [3.7] we have that every strong NQ-Hypervector space is a weak NQ-Hypervector space. Then by applying 3 of [3.20] the proof follows.

Proposition 3.22. *Let $U_1[T, I, F], U_2[T, I, F], \dots, U_n[T, I, F]$ be NQ-Hypersubspace of a super strong[strong] NQ-Hypervector space $V(T, I, F)$ over a neutrosophic field $K(T, I, F)$ (resp. $[K(I)]$). Then $\bigcap_{i=1}^n U_i$ is a NQ-Hypersubspace of $V(T, I, K)$.*

Proof: Same as in classical case.

Example 3.23. Let $M_1[T, I, F] = K(T, I, F) \times \{(0, 0T, 0I, 0F)\} \subseteq V(T, I, F)$ and $M_2[T, I, F] = \{(0, 0T, 0I, 0F)\} \times K(T, I, F) \subseteq V(T, I, F)$.

Following the approach in Example [3.19] we can establish that $M_1[T, I, F]$ and $M_2[T, I, F]$ are NQ-Hypersubspaces of $V(T, I, F)$.

Let $((a, bT, cI, dF), (0, 0T, 0I, 0F)) \in M_1[T, I, F]$ and $((0, 0T, 0I, 0F), (e, fT, gI, hF)) \in M_2[T, I, F]$.

Then

$$((a, bT, cI, dF), (0, 0T, 0I, 0F)) + ((0, 0T, 0I, 0F), (e, fT, gI, hF)) = (((a + 0), (b + 0)T, (c + 0)I, (d +$$

$0)F), ((0+e), (0+f)T, (0+g)I, (0+h)F)) = ((a, bT, cI, dF), (e, fT, gI, hF))$

But $\{((a, bT, cI, dF), (e, fT, gI, hF))\}$ is not a NQ-subset of $M_1[T, I, F] \cup M_2[T, I, F]$.

Therefore $M_1[T, I, F] \cup M_2[T, I, F]$ is not a NQ-Hypersubspace of $V(T, I, F)$.

This observation is recorded in the following remark.

Remark 3.24. Let $M_1[T, I, F]$ and $M_2[T, I, F]$ be NQ-Hypersubspaces of a super strong NQ-Hypervector space $V(T, I, F)$ over a NQ field $K(T, I, F)$, then generally, The union of two NQ-Hypersubspaces of a super strong NQ-Hypervector space $V(T, I, F)$ is not necessarily a NQ-Hypersubspace of $V(T, I, F)$.

Definition 3.25. Let $N_1[T, I, F]$ and $N_2[T, I, F]$ be any two NQ-Hypersubspaces of a super strong NQ-Hypervector space $V(T, I, F)$ over a NQ field $K(T, I, F)$ then the sum of $N_1[T, I, F]$ and $N_2[T, I, F]$ denoted by $N_1[T, I, F] + N_2[T, I, F]$ is called NQ Hyperlinear sum or NQ linear sum of the NQ-Hypersubspaces $N_1[T, I, F]$ and $N_2[T, I, F]$. And it is defined by the set

$$\bigcup \{n_1 + n_2 : n_1 = (a_1, b_1T, c_1I, d_1F) \in N_1[T, I, F], n_2 = (a_2, b_2T, c_2I, d_2F) \in N_2[T, I, F]\}$$

The NQ Hyperlinear sum of $N_1[T, I, F]$ and $N_2[T, I, F]$ is called the direct sum of the NQ-Hypersubspaces $N_1[T, I, F]$ and $N_2[T, I, F]$ if $N_1[T, I, F] \cap N_2[T, I, F] = \{\theta\}$.

Proposition 3.26. Let $N_1[T, I, F]$ and $N_2[T, I, F]$ be any two NQ-Hypersubspaces of a super strong NQ-Hypervector space $V(T, I, F)$ over a NQ field $K(T, I, F)$. Then

1. NQ Hyperlinear sum of $N_1[T, I, F]$ and $N_2[T, I, F]$ is a NQ-Hypersubspace of $V(T, I, F)$.
2. NQ Hyperlinear sum of $N_1[T, I, F]$ and $N_2[T, I, F]$ is the least NQ-Hypersubspace of $V(T, I, F)$ containing $N_1[T, I, F]$ and $N_2[T, I, F]$.

Proof:

1. Since $\theta = (0, 0T, 0I, 0F) \in N_1[T, I, F]$ and $\theta = (0, 0T, 0I, 0F) \in N_2[T, I, F]$,

then $\{\theta + \theta\} \subseteq N_1[T, I, F] + N_2[T, I, F]$

$\implies \{\theta\} \subseteq N_1[T, I, F] + N_2[T, I, F] \implies \theta \in N_1[T, I, F] + N_2[T, I, F]$,

therefore $N_1[T, I, F] + N_2[T, I, F]$ is non-empty.

Let $u = (a, bT, cI, dF), v = (e, fT, gI, hF) \in N_1[T, I, F] + N_2[T, I, F]$, then \exists

$u_1 = (a_1, b_1T, c_1I, d_1F), u_2 = (a_2, b_2T, c_2I, d_2F) \in N_1[T, I, F]$ and $v_1 = (e_1, f_1T, g_1I, h_1F)$

$v_2 = (e_2, f_2T, g_2I, h_2F) \in N_2[T, I, F]$ such that $u \in u_1 + v_1$ and $v \in u_2 + v_2$.

Let $\alpha = (p, qT, rI, sF), \beta = (p', q'T, r'I, s'F) \in K(T, I, F)$.

Then $\alpha \bullet u + \beta \bullet v \subseteq \alpha \bullet (u_1 + v_1) + \beta \bullet (u_2 + v_2)$

$= (p, qT, rI, sF) \bullet ((a_1 + e_1), (b_1 + f_1)T, (c_1 + g_1)I, (d_1 + h_1)F) + (p', q'T, r'I, s'F) \bullet ((a_2 + e_2), (b_2 + f_2)T, (c_2 + g_2)I, (d_2 + h_2)F)$

$\subseteq \{(x_1, y_1T, z_1I, w_1F) : x_1 \in p \bullet (a_1 + e_1), y_1 \in p \bullet (b_1 + f_1) \cup q \bullet (a_1 + e_1) \cup q \bullet (b_1 + f_1), z_1 \in p \bullet (c_1 + g_1) \cup q \bullet (c_1 + g_1) \cup r \bullet (a_1 + e_1) \cup r \bullet (b_1 + f_1) \cup r \bullet (c_1 + g_1), w_1 \in p \bullet (d_1 + h_1) \cup q \bullet (d_1 + h_1) \cup r \bullet (d_1 + h_1) \cup s \bullet (a_1 + e_1) \cup s \bullet (b_1 + f_1) \cup s \bullet (c_1 + g_1) \cup s \bullet (d_1 + h_1)\}$

$+ \{(x_2, y_2T, z_2I, w_2F) : x_2 \in p' \bullet (a_2 + e_2), y_2 \in p' \bullet (b_2 + f_2) \cup q' \bullet (a_2 + e_2) \cup q' \bullet (b_2 + f_2), z_2 \in p' \bullet (c_2 + g_2) \cup q' \bullet (c_2 + g_2) \cup r' \bullet (a_2 + e_2) \cup r' \bullet (b_2 + f_2) \cup r' \bullet (c_2 + g_2), w_2 \in p' \bullet (d_2 + h_2) \cup q' \bullet (d_2 + h_2) \cup r' \bullet (d_2 + h_2) \cup s' \bullet (a_2 + e_2) \cup s' \bullet (b_2 + f_2) \cup s' \bullet (c_2 + g_2) \cup s' \bullet (d_2 + h_2)\}$

$= \{(x, yT, zI, wF) : x \in (p \bullet a_1 + p \bullet e_1 + p' \bullet a_2 + p' \bullet e_2), y \in (p \bullet b_1 + p \bullet f_1 + p' \bullet b_2 + p' \bullet f_2) \cup (q \bullet a_1 + q \bullet e_1 + q' \bullet a_2 + q' \bullet e_2) \cup (q \bullet b_1 + q \bullet f_1 + q' \bullet b_2 + q' \bullet f_2),$

$z \in (p \bullet c_1 + p \bullet g_1 + p' \bullet c_2 + p' \bullet g_2) \cup (q \bullet c_1 + q \bullet g_1 + q' \bullet c_2 + q' \bullet g_2) \cup (r \bullet a_1 + r \bullet e_1 + r' \bullet a_2 + r' \bullet e_2) \cup (r \bullet b_1 + r \bullet f_1 + r' \bullet b_2 + r' \bullet f_2) \cup (r \bullet c_1 + r \bullet g_1 + r' \bullet c_2 + r' \bullet g_2)$

$w \in (p \bullet d_1 + p \bullet h_1 + p' \bullet d_2 + p' \bullet h_2) \cup (q \bullet d_1 + q \bullet h_1 + q' \bullet d_2 + q' \bullet h_2) \cup (r \bullet d_1 + r \bullet h_1 + r' \bullet d_2 + r' \bullet h_2) \cup (s \bullet a_1 + s \bullet e_1 + s' \bullet a_2 + s' \bullet e_2) \cup (s \bullet b_1 + s \bullet f_1 + s' \bullet b_2 + s' \bullet f_2) \cup (s \bullet c_1 + s \bullet g_1 + s' \bullet c_2 + s' \bullet g_2) \cup (s \bullet d_1 + s \bullet h_1 + s' \bullet d_2 + s' \bullet h_2)\}$

$= \{(k_1, m_1T, n_1I, j_1F) : k_1 \in (p \bullet a_1 + p' \bullet a_2), m_1 \in (p \bullet b_1 + p' \bullet b_2) \cup (q \bullet a_1 + q' \bullet a_2) \cup (q \bullet b_1 + q' \bullet b_2), n_1 \in (p \bullet c_1 + p' \bullet c_2) \cup (q \bullet c_1 + q' \bullet c_2) \cup (r \bullet a_1 + r' \bullet a_2) \cup (r \bullet b_1 + r' \bullet b_2) \cup (r \bullet c_1 + r' \bullet c_2), j_1 \in (p \bullet d_1 + p' \bullet d_2) \cup (q \bullet d_1 + q' \bullet d_2) \cup (r \bullet d_1 + r' \bullet d_2) \cup (s \bullet a_1 + s' \bullet a_2) \cup (s \bullet b_1 + s' \bullet b_2) \cup (s \bullet c_1 + s' \bullet c_2) \cup (s \bullet d_1 + s' \bullet d_2)\}$

$+ \{(k_2, m_2T, n_2I, j_2F) : k_2 \in (p \bullet e_1 + p' \bullet e_2), m_2 \in (p \bullet f_1 + p' \bullet f_2) \cup (q \bullet e_1 + q' \bullet e_2) \cup (q \bullet f_1 + q' \bullet f_2), n_2 \in (p \bullet g_1 + p' \bullet g_2) \cup (q \bullet g_1 + q' \bullet g_2) \cup (r \bullet e_1 + r' \bullet e_2) \cup (r \bullet f_1 + r' \bullet f_2) \cup (r \bullet g_1 + r' \bullet g_2), j_2 \in (p \bullet h_1 + p' \bullet h_2) \cup (q \bullet h_1 + q' \bullet h_2) \cup (r \bullet h_1 + r' \bullet h_2) \cup (s \bullet e_1 + s' \bullet e_2) \cup (s \bullet f_1 + s' \bullet f_2) \cup$

$$(s \bullet g_1 + s' \bullet g_2) \cup (s \bullet h_1 + s' \bullet h_2) \subseteq N_1[T, I, F] + N_2[T, I, F].$$

Hence $\alpha \bullet u + \beta \bullet v \subseteq N_1[T, I, F] + N_2[T, I, F]$.

Now since N_1, N_2 are proper subsets of $N_1[T, I, F]$ and $N_2[T, I, F]$ respectively, with both N_1 and N_2 being hypervector spaces. Then $N_1 + N_2$ is a hypervector space which is properly contained in $N_1[T, I, F] + N_2[T, I, F]$. Then we can conclude that $N_1[T, I, F] + N_2[T, I, F]$ is a NQ-Hypersubspace.

2. Let $N[T, I, F]$ be NQ-Hypersubspace of $V(T, I, F)$ such that $N_1[T, I, F] \subseteq N[T, I, F]$ and $N_2[T, I, F] \subseteq N[T, I, F]$.

Let $u = (a, bT, cI, dF) \in N_1[T, I, F] + N_2[T, I, F]$, then $\exists u_1 = (a_1, b_1T, c_1I, d_1F) \in N_1[T, I, F]$ and $u_2 = (a_2, b_2T, c_2I, d_2F) \in N_2[T, I, F]$ such that $u \in u_1 + u_2$.

Since $N_1[T, I, F] \subseteq N[T, I, F]$ and $N_2[T, I, F] \subseteq N[T, I, F]$, then $u_1, u_2 \in N[T, I, F]$.

Again since $N[T, I, F]$ is a NQ-Hypersubspace of $V(T, I, F)$, then we have that

$$u_1 + u_2 \subseteq N[T, I, F] \implies u \in N[T, I, F].$$

Hence $N_1[T, I, F] + N_2[T, I, F] \subseteq N[T, I, F]$ and the proof follows.

Proposition 3.27. Let $V(T, I, F)$ be a super strong NQ-Hypervector space over a NQ-field $K(T, I, F)$, let $u_1 = (a_1, b_1T, c_1I, d_1F), u_2 = (a_2, b_2T, c_2I, d_2F), \dots, u_n = (a_n, b_nT, c_nI, d_nF) \in V(T, I, F)$ and $\alpha_1 = (k_1, m_1T, r_1I, t_1F), \alpha_2 = (k_2, m_2T, r_2I, t_2F), \dots, \alpha_n = (k_n, m_nT, r_nI, t_nF) \in K(T, I, F)$. Then

1. $N(T, I, F) = \bigcup \{ \alpha_1 \bullet u_1 + \alpha_2 \bullet u_2 + \dots + \alpha_n \bullet u_n : \alpha_1, \alpha_2, \dots, \alpha_n \in K(T, I, F) \}$ is a NQ-Hypersubspace of $V(T, I, F)$.
2. $N(T, I, F)$ is the smallest NQ-Hypersubspace of $V(T, I, F)$ containing u_1, u_2, \dots, u_n .

Proof:

1. Follow similar approach as that of proposition 3.26 above.

2. Suppose that $H(T, I, F)$ is a super strong NQ-Hypersubspace of $V(T, I, F)$ containing

$$u_1 = (a_1, b_1T, c_1I, d_1F), u_2 = (a_2, b_2T, c_2I, d_2F), \dots, u_n = (a_n, b_nT, c_nI, d_nF).$$

Let $t \in N(T, I, F)$, then there exists $\alpha_1 = (k_1, m_1T, p_1I, q_1F), \alpha_2 = (k_2, m_2T, p_2I, q_2F), \dots,$

$$\alpha_n = (k_n, m_nT, p_nI, q_nF) \in K(T, I, F) \text{ such that}$$

$$t \in \alpha_1 \bullet (a_1, b_1T, c_1I, d_1F) + \alpha_2 \bullet (a_2, b_2T, c_2I, d_2F) + \dots + \alpha_n \bullet (a_n, b_nT, c_nI, d_nF) \subseteq H(T, I, F)$$

Therefore $t \in H(T, I, F) \implies N(T, I, F) \subseteq H(T, I, F)$.

Hence $N(T, I, F)$ is the smallest NQ-Hypersubspace of $V(T, I, F)$ containing u_1, u_2, \dots, u_n .

Note: The NQ-Hypersubspace $N(T, I, F)$ of the super strong NQ-Hypervector space $V(T, I, F)$ over a NQ field $K(T, I, F)$ of proposition 3.27 is said to be generated or spanned by the NQ-Hypervectors u_1, u_2, \dots, u_n and we write $N(T, I, F) = \text{span}\{u_1, u_2, \dots, u_n\}$.

Definition 3.28. Let $N_1[T, I, F]$ and $N_2[T, I, F]$ be two NQ-Hypersubspaces of a super strong NQ-Hypervector space $(V(T, I, F), +, \bullet, K(T, I, F))$ over a NQ field $K(T, I, F)$. $V(T, I, F)$ is said to be the direct sum of $N_1[T, I, F]$ and $N_2[T, I, F]$ written $V(T, I, F) = N_1[T, I, F] \oplus N_2[T, I, F]$ if every element $v \in V(T, I, F)$ can be written uniquely as $v = n_1 + n_2$ where $n_1 \in N_1[T, I, F]$ and $n_2 \in N_2[T, I, F]$.

Proposition 3.29. Let $N_1[T, I, F]$ and $N_2[T, I, F]$ be two NQ-Hypersubspaces of a super strong NQ-Hypervector space $(V(T, I, F), +, \bullet, K(T, I, F))$ over a NQ field $K(T, I, F)$. $V(T, I, F) = N_1[T, I, F] \oplus N_2[T, I, F]$ if and only if the following conditions hold:

1. $V(T, I, F) = N_1[T, I, F] + N_2[T, I, F]$.
2. $N_1[T, I, F] \cap N_2[T, I, F] = \{\theta\}$.

Proof : Same as in classical case.

Example 3.30. Let $V(T, I, F) = \mathbb{R}^3(T, I, F)$ be a super strong NQ-Hypervector space over a NQ-field $R(T, I, F)$ and let

$$N_1(T, I, F) = \{(u, \theta, w) : u = (a, bT, cI, dF), w = (k, mT, nI, pF) \in R(T, I, F)\} \text{ and}$$

$$N_2(T, I, F) = \{(\theta, v, \theta) : v = (e, fT, gI, hF) \in R(T, I, F)\}, \text{ be super strong NQ-Hypersubspaces of } V(T, I, F). \text{ Then } V(T, I, F) = N_1(T, I, F) \oplus N_2(T, I, F).$$

To see this, let $x = (u, v, w) \in V(T, I, F)$, then $x = (u, \theta, w) + (\theta, v, \theta)$, so $x \in N_1(T, I, F) + N_2(T, I, F)$. Hence $V(T, I, F) = N_1(T, I, F) + N_2(T, I, F)$. To show that $N_1(T, I, F) \cap N_2(T, I, F) = \{\theta\}$, let $x = (u, v, w) \in N_1(T, I, F) \cap N_2(T, I, F)$. Then $v = \theta$, i.e. $(e, fT, gI, hF) = (0, 0T, 0I, 0F)$ because x lies in $N_1(T, I, F)$, and $u = w = \theta$ i.e. $(a, bT, cI, dF) = (k, mT, nI, pF) = (0, 0T, 0I, 0F)$ because x lies in $N_2(T, I, F)$. Thus $x = (\theta, \theta, \theta) = \theta$, so $\theta = (0, 0T, 0I, 0F)$ is the only NQ-Hypervector in $N_1(T, I, F) \cap N_2(T, I, F)$. Hence $N_1(T, I, F) \cap N_2(T, I, F) = \{0, 0T, 0I, 0F\} = \{\theta\}$.
 $\implies V(T, I, F) = N_1(T, I, F) \oplus N_2(T, I, F)$.

Definition 3.31. Let $N[T, I, F]$ be a NQ-Hypersubspace of a super strong NQ-Hypervector space $(V(T, I, F), +, \bullet, K(T, I, F))$ over a NQ-field $K(T, I, F)$. The quotient $V(T, I, F)/N[T, I, F]$ is defined by the set

$$\{[v] = v + N[T, I, F] : v \in V(T, I, F)\}.$$

If for every $[u], [v] \in V(T, I, F)/N[T, I, F]$ and $\alpha \in K(T, I, F)$, we define:

$$[u] \oplus [v] = (u + v) + N[T, I, F]$$

and

$$\alpha \odot [u] = [\alpha \bullet u] = \{[x] : x \in \alpha \bullet u\},$$

it can be shown that $(V(T, I, F)/N[T, I, F], \oplus, \odot, K(T, I, F))$ is a super strong NQ-Hypervector space over NQ-field $K(T, I, F)$ called a super strong NQ quotient hypervector space.

4 Linear Dependence, Independence, Bases and Dimensions of NQ-Hypervector Space

Definition 4.1. Let $(V(T, I, F), +, \bullet, K(T, I, F))$ be a super strong NQ-Hypervector space over a NQ field $K(T, I, F)$ and let

$B(T, I, F) = \{u_1 = (a_1, b_1T, c_1I, d_1F), u_2 = (a_2, b_2T, c_2I, d_2F), \dots, u_n = (a_n, b_nT, c_nI, d_nF)\}$ be a subset of $V(T, I, F)$. $B(T, I, F)$ is said to generate or span $V(T, I, F)$ if $V(T, I, F) = \text{span}(B(T, I, F))$.

Example 4.2. Let $V(T, I, F) = \mathbb{R}^4(T, I, K)$ be a super strong NQ-Hypervector space over a NQ field $R(T, I, F)$ and let $B(T, I, F) = \{u_1 = ((1, 0T, 0I, 0F), (0, 0T, 0I, 0F), (0, 0T, 0I, 0F), (0, 0T, 0I, 0F)), u_2 = ((0, 0T, 0I, 0F), (1, 0T, 0I, 0F), (0, 0T, 0I, 0F), (0, 0T, 0I, 0F)), u_3 = ((0, 0T, 0I, 0F), (0, 0T, 0I, 0F), (1, 0T, 0I, 0F), (0, 0T, 0I, 0F)), u_4 = ((0, 0T, 0I, 0F), (0, 0T, 0I, 0F), (0, 0T, 0I, 0F), (1, 0T, 0I, 0F))\}$. Then $B(T, I, F)$ spans $V(T, I, F)$.

Definition 4.3. Let $(V(T, I, F), +, \bullet, K(T, I, F))$ be a super strong NQ-Hypervector space over NQ-field $K(T, I, F)$. The NQ vector $u = (a, bT, cI, dF) \in V(T, I, F)$ is said to be a linear combination of the NQ vectors $u_1 = (a_1, b_1T, c_1I, d_1F), u_2 = (a_2, b_2T, c_2I, d_2F), \dots, u_n = (a_n, b_nT, c_nI, d_nF) \in V(I_1, I_2)$ if there exists NQ-scalars $\alpha_1 = (k_1, m_1T, s_1I, t_1F), \alpha_2 = (k_2, m_2T, s_2I, t_2F), \dots, \alpha_n = (k_n, m_nT, s_nI, t_nF) \in K(T, I, F)$ such that

$$u \in \alpha_1 \bullet u_1 + \alpha_2 \bullet u_2 + \dots + \alpha_n \bullet u_n.$$

Example 4.4. Let $V(T, I, F) = \mathbb{R}(T, I, F)$ be a weak NQ-Hypervector space over a field $K = \mathbb{R}$. An element $v = (1, 1T, 4I, 7F) \in V(T, I, F)$ is a linear combination of the elements $v_1 = (1, 2T, -1I, -2F), v_2 = (3, 5T, 2I, 3F) \in V(T, I, F)$

Since

$$(1, 1T, 4I, 7F) \in -2 \bullet (1, 2T, -1I, -2F) + 1 \bullet (3, 5T, 2I, 3F).$$

Definition 4.5. Let $(V(T, I, F), +, \bullet, K(T, I, F))$ be a super strong NQ-Hypervector space over a NQ field $K(T, I, F)$ and let

$B(T, I, F) = \{u_1 = (a_1, b_1T, c_1I, d_1F), u_2 = (a_2, b_2T, c_2I, d_2F), \dots, u_n = (a_n, b_nT, c_nI, d_nF)\}$ be a subset of $V(T, I, F)$.

1. $B(T, I, F)$ is called a linearly dependent set if there exists NQ scalars $\alpha_1 = (k_1, m_1T, s_1I, t_1F), \alpha_2 = (k_2, m_2T, s_2I, t_2F), \dots, \alpha_n = (k_n, m_nT, s_nI, t_nF)$ (not all zero) such that

$$\theta \in \alpha_1 \bullet u_1 + \alpha_2 \bullet u_2 + \dots + \alpha_n \bullet u_n.$$

2. $B(T, I, F)$ is called a linearly independent set if

$$\theta \in \alpha_1 \bullet u_1 + \alpha_2 \bullet u_2 + \cdots + \alpha_n \bullet u_n$$

implies that $\alpha_1 = \alpha_2 = \cdots = \alpha_n = (0, 0T, 0I, 0F) = \theta$

Example 4.6. Let $V(T, I, F) = \mathbb{R}(T, I, F)$ be a weak NQ-Hypervector space over a field $K = \mathbb{R}$. The subset $B(T, I, F) = \{(5, -7T, 5I, 4F), (3, -4T, 2I, 2F), (-2, 3T, -3T, -2T)\}$ of $V(T, I, F)$ is NQ linearly dependent set since

$$\theta \in 1 \bullet (5, -7T, 5I, 4F) + (-1) \bullet (3, -4T, 2I, 2F) + 1 \bullet (-2, 3T, -3T, -2T)$$

Example 4.7. Let $V(T, I, F) = \mathbb{R}(T, I, F)$ be a weak NQ-Hypervector space over a field $K = \mathbb{R}$. The subset $B(T, I, F) = \{(7, 0T, 0I, 0F), (0, 3T, 5I, 0F), (0, 0T, 0T, -8T)\}$ of $V(T, I, F)$ is NQ linearly independent set over \mathbb{R} because we can not find $a, b, c \in \mathbb{R}$ such that

$$\theta \in a \bullet (7, 0T, 0I, 0F) + b \bullet (0, 3T, 5I, 0F) + c \bullet (0, 0T, 0T, -8T)$$

If possible then $\theta \in a \bullet (7, 0T, 0I, 0F) + b \bullet (0, 3T, 5I, 0F) + c \bullet (0, 0T, 0T, -8T)$ implies that;

$$0 \in a \bullet 7 + b \bullet 0 + c \bullet 0 \text{ which forces } a = 0,$$

$$0 \in a \bullet 0 + b \bullet 3 + c \bullet 0 \text{ which forces } b = 0,$$

$$0 \in a \bullet 0 + b \bullet 5 + c \bullet 0 \text{ which forces } b = 0 \text{ and}$$

$$0 \in a \bullet 0 + b \bullet 0 + c \bullet -8 \text{ which forces } c = 0.$$

Thus the equations are consistent and $a = b = c = 0$.

Proposition 4.8. Let $(V(T, I, F), +, \bullet, K)$ be a weak NQ-Hypervector space over a field K . Any singleton set of non-null NQ vector of the weak NQ-Hypervector space $V(T, I, F)$ is linearly independent.

Proof: Suppose that $\theta \neq v = (a, bT, cI, dF) \in V(T, I, F)$. Let $\theta \in k \bullet v$ and suppose that $\theta \neq k \in K$. Then $k^{-1} \in K$ and therefore, $k^{-1} \bullet \theta \subseteq k^{-1} \bullet (k \bullet v)$ so that

$$\begin{aligned} \theta &\in (k^{-1}k) \bullet v \\ &= 1 \bullet v \\ &= \{(x, yT, zI, wF) : x \in 1 \bullet a, y \in 1 \bullet b, z \in 1 \bullet c, w \in 1 \bullet d\} \\ &= \{(x, yT, zI, wF) : x \in \{a\}, y \in \{b\}, z \in \{c\}, w \in \{d\}\} \\ &= \{(a, bT, cI, dF)\} \\ &= \{v\} \end{aligned}$$

This shows that $v = \theta$ which is a contradiction. Hence, $k = \theta$ and thus, the singleton $\{v\}$ is a linearly independent set.

We note that the singleton set will be linearly dependent if it contains a null NQ-vector and $\theta \neq k \in K$. This observation is recorded in the next proposition.

Proposition 4.9. Let $(V(T, I, F), +, \bullet, K)$ be a weak NQ-Hypervector space over a field K . Any set of NQ-vectors of the weak NQ-Hypervector space $V(T, I, F)$ containing the null NQ-vector is always linearly dependent.

Proof: Follows from Proposition 4.8

Proposition 4.10. Let $(V(T, I, F), +, \bullet, K)$ be a weak NQ-Hypervector space over a field K and let

$$B(I_1, I_2) = \{u_1 = (a_1, b_1T, c_1I, d_1F), u_2 = (a_2, b_2T, c_2I, d_2F), \dots, u_n = (a_n, b_nT, c_nI, d_nF)\}$$

be a subset of $V(T, I, F)$. Then $B(T, I, F)$ is a linearly dependent set if and only if at least one element of $B(T, I, F)$ can be expressed as a linear combination of the remaining elements of $B(T, I, F)$.

Proof : Suppose that $B(T, I, F)$ is a linearly dependent set. Then there exists scalars k_1, k_2, \dots, k_n not all zero in K such that

$$\theta \in k_1 \bullet u_1 + k_2 \bullet u_2 + \cdots + k_n \bullet u_n.$$

Suppose that $k_1 \neq 0$, then $k_1^{-1} \in K$ and therefore

$$\begin{aligned} k_1^{-1} \bullet \theta &\subseteq k_1^{-1} \bullet (k_1 \bullet u_1 + k_2 \bullet u_2 + \cdots + k_n \bullet u_n) \\ &= (k_1^{-1}k_1) \bullet u_1 + (k_1^{-1}k_2) \bullet u_2 + \cdots + (k_1^{-1}k_n) \bullet u_n \\ &= 1 \bullet u_1 + (k_1^{-1}k_2) \bullet u_2 + \cdots + (k_1^{-1}k_n) \bullet u_n \end{aligned}$$

This implies that

$$\begin{aligned} -u_1 &\in (k_1^{-1}k_2) \bullet u_2 + \cdots + (k_1^{-1}k_n) \bullet u_n \\ (u_1) &\in (-1) \bullet [(k_1^{-1}k_2) \bullet u_2 + \cdots + (k_1^{-1}k_n) \bullet u_n] \\ &\subseteq (-1) \bullet ((k_1^{-1}k_2) \bullet u_2 + \cdots + (-1) \bullet (k_n^{-1}k_n) \bullet u_n) \\ &\subseteq (-k_1^{-1}k_2) \bullet u_2 + (-k_1^{-1}k_3) \bullet u_3 + \cdots + (-k_1^{-1}k_n) \bullet u_n. \end{aligned}$$

This shows that $u_1 \in \text{span}\{u_2, u_3, \dots, u_n\}$.

Conversely, suppose that $u_1 \in \text{span}\{u_2, u_3, \dots, u_n\}$ and suppose that $0 \neq -1 \in K$. Then there exists $k_2, k_3, \dots, k_n \in K$ such that

$$u_1 \in k_2 \bullet u_2 + k_3 \bullet u_3 + \cdots + k_n \bullet u_n$$

and we have

$$u_1 + (-u_1) \in (-1) \bullet u_1 + k_2 \bullet u_2 + k_3 \bullet u_3 + \cdots + k_n \bullet u_n.$$

from which we have

$$\theta \in (-1) \bullet u_1 + k_2 \bullet u_2 + k_3 \bullet u_3 + \cdots + k_n \bullet u_n.$$

Since $-1 \neq 0 \in K$, it follows that $B(T, I, F)$ is a linearly dependent set.

Proposition 4.11. Let $(V(T, I, F), +, \bullet, K(T, I, F))$ be a super strong NQ-Hypervector space over a NQ-field $K(T, I, F)$ and let $M(T, I, F)$ and $N(T, I, F)$ be subsets of $V(T, I, F)$ such that $M(T, I, F) \subseteq N(T, I, F)$.

1. If $M(T, I, F)$ is linearly dependent, then $N(T, I, F)$ is linearly dependent.
2. If $N(T, I, F)$ is linearly independent, then $M(T, I, F)$ is linearly independent.

Proof: Same as in classical case.

Definition 4.12. Let $V(T, I, F)$ be a super strong(strong) NQ-Hypervector space over a NQ field $K(T, I, F)$ (resp. neutrosophic field $K(I)$) and let

$B(T, I, F) = \{u_1 = (a_1, b_1T, c_1I, d_1F), u_2 = (a_2, b_2T, c_2I, d_2F), \dots, u_n = (a_n, b_nT, c_nI, d_nF)\}$ be a subset of $V(T, I, F)$. $B(T, I, F)$ is said to be a basis for $V(T, I, F)$ if the following conditions hold:

1. $B(T, I, F)$ is a linearly independent set
2. $V(T, I, F) = \text{span}(B(T, I, F))$.

If $B(T, I, F)$ is finite and its cardinality is n , then $V(T, I, F)$ is called an n -dimensional super strong(strong) NQ-Hypervector space and we write $\dim_{ss}(V(T, I, F))$ (resp. $(\dim_s V(T, I, F))$) = n . If $B(T, I, F)$ is not finite, then $V(T, I, F)$ is called an infinite-dimensional super strong(strong) NQ-Hypervector space.

Example 4.13. In [4.2](#), $B(T, I, F)$ is a basis for $V(T, I, F)$ and $\dim_{ss} V(T, I, F) = 4$

Proposition 4.14. Let $(V(T, I, F), +, \bullet, K(T, I, F))$ be a finite dimensional super strong NQ-Hypervector space over a NQ field $K(T, I, F)$ and let

$B(T, I, F) = \{x_1 = (a_1, b_1T, c_1I, d_1F), x_2 = (a_2, b_2T, c_2I, d_2F), \dots, u_n = x_n = (a_n, b_nT, c_nI, d_nF)\}$ be a basis for $V(T, I, F)$. Then every non null NQ-Hypervector $x = (a, bT, cI, dF) \in V(T, I, K)$ has a unique representation.

Proof: Since $B(T, I, F)$ is a basis for $V(T, I, F)$ and $x \in V(T, I, K)$, there exist $\alpha_1 = (k_1, m_1T, n_1I, t_1F)$, $\alpha_2 = (k_2, m_2T, n_2I, t_2F), \dots, \alpha_n = (k_n, m_nT, n_nI, t_nF)$ such that

$$x \in \alpha_1 \bullet x_1 + \alpha_2 \bullet x_2 + \cdots + \alpha_n \bullet x_n \quad (1).$$

Suppose we also let $x \in \beta_1 \bullet x_1 + \beta_2 \bullet x_2 + \cdots + \beta_n \bullet x_n$, for some $\beta_1 = (p_1, q_1T, r_1I, s_1F)$,

$\beta_2 = (p_2, q_2T, r_2I, s_2F), \dots, \beta_n = (p_n, q_nT, r_nI, s_nF) \in K(T, I, F)$.

$$\begin{aligned} \text{Therefore, } -x &\in (-1) \bullet x \subseteq (-1) \bullet (\beta_1 \bullet x_1 + \beta_2 \bullet x_2 + \cdots + \beta_n \bullet x_n) \\ &\Rightarrow -x \in ((-1) \bullet (\beta_1 \bullet x_1)) + ((-1) \bullet (\beta_2 \bullet x_2)) + \cdots + ((-1) \bullet (\beta_n \bullet x_n)) \\ &= ((-1 \bullet \beta_1) \bullet x_1) + ((-1 \bullet \beta_2) \bullet x_2) + \cdots + ((-1 \bullet \beta_n) \bullet x_n) \\ &= (-\beta_1) \bullet x_1 + (-\beta_2) \bullet x_2 + \cdots + (-\beta_n) \bullet x_n. \end{aligned}$$

Therefore

$$-x \in (-\beta_1) \bullet x_1 + (-\beta_2) \bullet x_2 + \cdots + (-\beta_n) \bullet x_n \quad (2).$$

From (1) and (2) we obtain

$$x + (-x) \subseteq (\alpha_1 \bullet x_1 + \alpha_2 \bullet x_2 + \cdots + \alpha_n \bullet x_n) + ((-\beta_1) \bullet x_1 + (-\beta_2) \bullet x_2 + \cdots + (-\beta_n) \bullet x_n).$$

Therefore $\theta \in x + (-x) \subseteq (\alpha_1 + (-\beta_1)) \bullet x_1 + (\alpha_2 + (-\beta_2)) \bullet x_2 + \cdots + (\alpha_n + (-\beta_n)) \bullet x_n$.

Since $\{x_1, x_2, \dots, x_n\}$ is a basis for $V(T, I, F)$ and

$$\theta \in (\alpha_1 - \beta_1) \bullet x_1 + (\alpha_2 - \beta_2) \bullet x_2 + \cdots + (\alpha_n - \beta_n) \bullet x_n.$$

Then it follows that $\theta \in \alpha_i - \beta_i$, for all $i = 1, 2, \dots, n$. Hence $a_i = b_i$, for all $i = 1, 2, \dots, n$.

5 Conclusion

In this paper, we have studied Hypervector Space in the Neutrosophic Quadruple (NQ) environment. Their basic properties have been extended and established in the Neutrosophic Quadruple (NQ) environment. We hope to study the homomorphisms and establish more advanced properties of this structure in our future work.

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ON Neutrosophic Crisp Supra Bi-Topological Spaces

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Abstract

In this paper, neutrosophic crisp supra bi-topological structure, which is a more general structure than neutrosophic crisp supra topological spaces, is built on neutrosophic crisp sets. The necessary arguments which are pairwise neutrosophic crisp supra open set, pairwise neutrosophic crisp supra closed set, pairwise neutrosophic crisp supra closure, pairwise neutrosophic crisp supra interior is defined, and their basic properties are presented. Finally, many examples are presented.

Keywords: Neutrosophic crisp supra bi-topological spaces, neutrosophic crisp supra pairwise open (closed) sets, neutrosophic crisp supra bi-open (closed) sets.

1. Introduction

The concept of neutrosophy is a new branch of Philosophy introduced by Smarandache [1,2], and has many applications in different fields of sciences such as topology. As a generalization of the concept of topological spaces, Salama and Smarandache [3] defined neutrosophic crisp topological spaces in 2014. The crisp supra topological space was introduced by Mashhour et al. [4] In 1983, as a generalization of the concept of topological space. Jayaparthasarathy et al.[5] generalized this concept and introduced the concept of neutrosophic supra topological space in 2019, by using the neutrosophic fuzzy sets. Also, Al-Hamido presented a more general study, where he created the concept of neutrosophic crisp supra topological spaces [6] in 2020. In 2018 AL-Nafee et al. [7] introduced the notion of new neutrosophic crisp points and neutrosophic crisp separation axioms in neutrosophic crisp topological space. The concept of supra bi-topological spaces was introduced by Gowri, and Rajayal [8] as an extension of supra topological spaces in 2017. On the other hand, the concept of bi-topological spaces was introduced by Kelly [9] as an extension of topological spaces in 1963. He did define bi-topological space as a set endowed with two topologies. The concept of neutrosophic bi-topological spaces was introduced by Al-Hamido [10] as an extension of neutrosophic topological spaces in 2019. This concept has been studied in [11]. Also, the concept of neutrosophic crisp bi-

topological spaces was introduced by Al-Hamido [12] as an extension of neutrosophic crisp topological spaces in 2018. Since the discovery of neutrosophic topological space and neutrosophic crisp topological space, there has been a concerted research efforts to find new neutrosophic open sets and neutrosophic crisp open sets, for more detail see [13-31].

In this paper, we use the neutrosophic crisp sets to introduce neutrosophic crisp supra bi-topological space. Also, we introduce new class of neutrosophic crisp supra open (closed) sets in this space, as pairwise neutrosophic crisp supra open set, pairwise neutrosophic crisp supra closed set, pairwise neutrosophic crisp supra closure, and we study some basic properties of this new neutrosophic crisp supra open (closed) sets.

2. Preliminaries

In this part, we recall some basic definitions and properties which are useful in this paper.

Definition 2.1. [3]

Let $X \neq \emptyset$ be a fixed set. A neutrosophic crisp set (NCS) U is an object having the form $U = \langle U_1, U_2, U_3 \rangle$; U_1, U_2 and U_3 are subsets of X , satisfying $U_1 \cap U_2 = \emptyset, U_1 \cap U_3 = \emptyset$ and $U_2 \cap U_3 = \emptyset$.

Definition 2.2. [3]

\emptyset_N maybe defined in four ways as a neutrosophic crisp set, as follows :

1. $\emptyset_N = \langle \emptyset, \emptyset, X \rangle$.
2. $\emptyset_N = \langle \emptyset, X, \emptyset \rangle$.
3. $\emptyset_N = \langle \emptyset, X, X \rangle$.
4. $\emptyset_N = \langle \emptyset, \emptyset, \emptyset \rangle$.

X_N may be defined in four ways as a neutrosophic crisp set, as follows :

1. $X_N = \langle X, \emptyset, \emptyset \rangle$.
2. $X_N = \langle X, X, \emptyset \rangle$.
3. $X_N = \langle X, \emptyset, X \rangle$.
4. $X_N = \langle X, X, X \rangle$.

Definition 2.3. [3]

Let $X \neq \emptyset$ be a fixed set, and $U = \langle U_1, U_2, U_3 \rangle, V = \langle V_1, V_2, V_3 \rangle$ are two neutrosophic crisp sets, then:

$U \cup V$ may be defined as two ways, as follows :

1. $U \cup V = \langle U_1 \cup V_1, U_2 \cup V_2, U_3 \cap V_3 \rangle$.
2. $U \cup V = \langle U_1 \cup V_1, U_2 \cap V_2, U_3 \cap V_3 \rangle$.

$U \cap V$ may be defined as two ways, as follows :

3. $U \cap V = \langle U_1 \cap V_1, U_2 \cap V_2, U_3 \cup V_3 \rangle$.

$$4. \quad U \cap V = \langle U_1 \cap V_1, U_2 \cup V_2, U_3 \cup V_3 \rangle.$$

Definition 24. [3]

A neutrosophic crisp topology (NCT) on a non-empty set X is a family T of neutrosophic crisp subsets in X may be satisfying the following axioms:

1. X_N and \emptyset_N belong to T .
2. T is closed under finite intersection.
3. T is closed under arbitrary union.

The pair (X, T) is a neutrosophic crisp topological space (NCTS) in X . Moreover, the elements in T are said to be neutrosophic crisp open sets (NCOS). A neutrosophic crisp set F is closed (NCCS) if and only if its complement F^c is a neutrosophic crisp open set.

Definition 25. [6]

A neutrosophic crisp supra topology (NCST) on a non-empty set X is a family S of neutrosophic crisp subsets in X may be satisfying the following axioms:

1. X_N and \emptyset_N belong to S .
2. S is closed under arbitrary union.

The pair (X, S) is said to be a neutrosophic crisp supra topological space (NCSTS) in X . Moreover, the elements in S are said to be neutrosophic crisp supra open sets (NCSOS). A neutrosophic crisp set F is neutrosophic crisp supra closed (NCSCS) if and only if its complement F^c is neutrosophic crisp supra open.

3. Neutrosophic crisp supra bi-topological space

In this section, we introduce the neutrosophic crisp supra bi-topological space. Moreover, we introduce new types of neutrosophic crisp supra open (closed) sets in this space and study their properties.

Definition 3.1.

Let Γ_1, Γ_2 is two neutrosophic crisp supra topologies on a nonempty set X then (X, Γ_1, Γ_2) is a neutrosophic crisp supra bi-topological space (SBI-NCTS for short).

Example 3.2.

Let $X = \{a, b\}$, $\Gamma_1 = \{\emptyset_N, X_N, A, B, E\}$, $\Gamma_2 = \{\emptyset_N, X_N, B, G\}$; $A = \{\langle \{a\}, \emptyset, \emptyset \rangle\}$, $B = \{\langle \emptyset, \{b\}, \emptyset \rangle\}$, $E = \{\langle \{a\}, \{b\}, \emptyset \rangle\}$, $G = \{\langle \emptyset, \emptyset, \{a\} \rangle\}$.

Then $(X, \Gamma_1), (X, \Gamma_2)$ are neutrosophic crisp supra spaces. Therefore (X, Γ_1, Γ_2) is a neutrosophic crisp supra bi-topological space.

Definition 3.3.

Let (X, Γ_1, Γ_2) be a neutrosophic crisp supra bi-topological space. Elements in $\Gamma_1 \cup \Gamma_2$ are said to be neutrosophic crisp supra bi-open sets (SBI-NCOS for short). A neutrosophic crisp set F is neutrosophic crisp supra closed (SBI-NCCS for short) if and only if its complement F^c is a neutrosophic crisp supra bi-open set.

- The family of all neutrosophic crisp supra bi-open sets is denoted by (SBI-NCOS(X)).

- The family of all neutrosophic crisp supra bi-closed sets is denoted by (SBI-NCCS(X)).

Example 3.4.

In Example 3.2, the neutrosophic crisp supra bi-open sets (SBI-NCOS) are :

$$\text{SBI-NCOS}(X) = \{\emptyset_N, X_N, A, B, E, G\}.$$

Remark 3.5.

1. Every neutrosophic crisp supra open sets in (X, Γ_1) or (X, Γ_2) is a neutrosophic crisp supra bi-open set.
2. Every neutrosophic crisp supra closed sets in (X, Γ_1) or (X, Γ_2) is a neutrosophic crisp supra bi-closed set.

Remark 3.6.

Every neutrosophic crisp supra bi-topological space (X, Γ_1, Γ_2) induces two neutrosophic crisp supra topological spaces as $(X, \Gamma_1), (X, \Gamma_2)$.

Remark 3.7.

If (X, Γ) is neutrosophic crisp supra topological space. Then (X, Γ, Γ) is a neutrosophic crisp supra bi-topological space.

Remark 3.8.

Let (X, Γ_1, Γ_2) is a neutrosophic crisp supra bi-topological space (SBI-NCTS). Then the union of two neutrosophic crisp supra bi-open (bi-closed) sets is not necessary a neutrosophic crisp supra bi-open (bi-closed) set as the following example shows that.

Example 3.9.

Let $X = \{a, b\}$, $\Gamma_1 = \{\emptyset_N, X_N, A\}$, $\Gamma_2 = \{\emptyset_N, X_N, B\}$; $A = \{<\{a\}, \emptyset, \emptyset>\}$, $B = \{<\emptyset, \{b\}, \{a\}>\}$. Then (X, Γ_1, Γ_2) is a neutrosophic crisp supra bi-topological space.

A, B are two neutrosophic crisp supra bi-open sets but $A \cup B = \{<\{a\}, \{b\}, \emptyset>\}$ is not a neutrosophic crisp supra bi-open set.

Also, A^c, B^c are two neutrosophic crisp supra bi-closed sets but $A^c \cup B^c = \{<X, X, \{b\}>\}$ is not a neutrosophic crisp supra bi-closed set.

Remark 3.10.

Let (X, Γ_1, Γ_2) be a neutrosophic crisp supra bi-topological space (SBI-NCTS). Then the intersection of two neutrosophic crisp supra bi-open (bi-closed) sets is not necessary a neutrosophic crisp supra bi-open (bi-closed) set as the following example shows that.

Example 3.11.

In Example 3.9, A, B are two neutrosophic crisp supra bi-open sets, but $A \cap B = \{\langle \emptyset, \emptyset, \{a\} \rangle\}$ is not neutrosophic crisp supra bi-open set.

Also, A^c, B^c are two neutrosophic crisp supra bi-closed sets, but $A^c \cap B^c = \{\langle \{b\}, \{a\}, X \rangle\}$ is not neutrosophic crisp supra bi-closed set.

4. The interior and the closure via neutrosophic Supra bi-open (closed) sets

In this section we define the closure and interior neutrosophic crisp supra set based on these new varieties of neutrosophic crisp supra open and closed sets. Also, we introduce the basic properties of closure and the interior.

Definition 4.1.

Let (X, Γ_1, Γ_2) be a neutrosophic crisp supra bi-topological space, and A be a neutrosophic crisp supra set then :

The union of any neutrosophic crisp supra bi-open sets contained in A is called a neutrosophic crisp supra bi-interior of A ($NCSint(A)$).

$$NCSint(A) = \cup \{B ; B \subseteq A; B \in SBi-NCOS(X)\}.$$

Theorem 4.2.

Let (X, Γ_1, Γ_2) be a neutrosophic crisp supra bi-topological space. If A, B are neutrosophic crisp supra sets then :

1. $NCSint(A) \subseteq A$.
2. $NCSint(A)$ is not necessary a neutrosophic crisp supra bi-open set.
3. $A \subseteq B \Rightarrow NCSint(A) \subseteq NCSint(B)$.

Proof :

1. The proof follows from the definition of $NCSint(A)$ as a union of any neutrosophic crisp supra bi-open sets contained in A .
2. The proof follows from Remark 3.8.
3. Obvious.

Definition 4.3.

Let (X, Γ_1, Γ_2) be a neutrosophic crisp supra bi-topological space. If A is neutrosophic crisp supra set then :

The intersection of any a neutrosophic crisp supra bi-closed sets containing A is called neutrosophic crisp supra

bi-closure of A ($\text{NCscl}(A)$).

$$\text{NCscl}(A) = \bigcap \{B : B \supseteq A; B \in \text{SBI-NCCS}(X)\}.$$

Theorem 4.4.

Let (X, Γ_1, Γ_2) be a neutrosophic crisp supra bi-topological space and A be a neutrosophic crisp supra set then:

1. $A \subseteq \text{NCscl}(A)$.
2. $\text{NCscl}(A)$ is not necessary a neutrosophic crisp supra bi-closed set.

Proof :

1. The proof follows from the definition of $\text{NCscl}(A)$ as an intersection of any neutrosophic crisp supra bi-closed set contained A .
2. The proof follows from Remark 3.10.

5. Pairwise neutrosophic crisp supra open (closed) sets

In this section, we introduce new concept of open and closed sets in neutrosophic crisp supra bi-topological space, as a pairwise neutrosophic crisp supra open(closed) sets. Also, we investigate the basic properties of this new concept of open and closed sets in SBI-NCTS .

Definition 5.1

Let (X, Γ_1, Γ_2) be a neutrosophic crisp supra bi-topological space. A neutrosophic crisp supra set A over X is said to be a pairwise neutrosophic crisp supra open set in (X, Γ_1, Γ_2) if there exists a neutrosophic crisp supra open set B in Γ_1 and a neutrosophic crisp supra open set C in Γ_2 such that $A = B \cup C$.

Definition 5.2

Let (X, Γ_1, Γ_2) be a neutrosophic crisp supra bi-topological space. A neutrosophic crisp supra set A over X is said to be a pairwise neutrosophic crisp supra closed set in (X, Γ_1, Γ_2) if its crisp neutrosophic complement is a pairwise neutrosophic crisp supra open set in (X, Γ_1, Γ_2) . Obviously, a neutrosophic crisp set A over X is a pairwise neutrosophic crisp supra closed set in (X, Γ_1, Γ_2) if there exists a neutrosophic crisp supra closed set B in $(\Gamma_1)^c$ and a neutrosophic crisp supra closed set C in $(\Gamma_2)^c$ such that $A = B \cap C$.

The family of all pairwise neutrosophic crisp supra open (closed) sets in (X, Γ_1, Γ_2) is denoted by $\text{PNCsO}(X, \Gamma_1, \Gamma_2)$ [$\text{PNCsC}(X, \Gamma_1, \Gamma_2)$].

Example 5.3

In Example 3.9, the family of all pairwise neutrosophic crisp supra open (closed) sets in (X, Γ_1, Γ_2)

$$\text{PNC SO}(X, \Gamma_1, \Gamma_2) = \{ \emptyset_N, X_N, A, B, A \cup B \};$$

$$A \cup B = \{ \langle \{a\}, \{b\}, \emptyset \rangle \}.$$

Theorem 5.4

Let (X, Γ_1, Γ_2) be a neutrosophic crisp supra bi-topological space. Then the family of all pairwise neutrosophic crisp supra open set is a neutrosophic crisp supra topology on X . This neutrosophic crisp supra topology is denoted by Γ_{12} .

Proof:

Since $\emptyset_N \cup \emptyset_N = \emptyset_N$, hence $\emptyset_N \in \Gamma_1, \Gamma_2$. Therefore $\emptyset_N \in \text{PNC SO}(X, \Gamma_1, \Gamma_2)$. Similarly, $X_N \in \text{PNC SO}(X, \Gamma_1, \Gamma_2)$.

Let $\{(N_i): i \in I\} \subseteq \text{PNC SO}(X, \Gamma_1, \Gamma_2)$. N_i is a pairwise neutrosophic crisp supra open set, $\forall i \in I$.

There exist $N_{i1} \in \Gamma_1$ and $N_{i2} \in \Gamma_2$ such that $N_i = N_{i1} \cup N_{i2} \forall i \in I$, which implies that:

$$\bigcup_{i \in I} N_i = \bigcup_{i \in I} (N_{i1} \cup N_{i2}) = (\bigcup_{i \in I} [N_{i1}]) \cup (\bigcup_{i \in I} [N_{i2}]).$$

Since Γ_1, Γ_2 are neutrosophic crisp supra topologies, $\bigcup_{i \in I} [N_{i1}] \in \Gamma_1$ and $\bigcup_{i \in I} [N_{i2}] \in \Gamma_2$.

Therefore $\bigcup_{i \in I} N_i$ is a pairwise neutrosophic crisp supra open set.

Remark 5.5

Let (X, Γ_1, Γ_2) be a neutrosophic crisp supra bi-topological space. Then an arbitrary intersection of pairwise neutrosophic crisp supra closed sets is a pairwise neutrosophic crisp supra closed set.

Proof:

Let $\{(N_i): i \in I\} \subseteq \text{PNC SC}(X, \Gamma_1, \Gamma_2)$. Then N_i is a pairwise neutrosophic crisp supra closed set $\forall i \in I$,

therefore there exist $N_{i1} \in (\Gamma_1)^c$ and $N_{i2} \in (\Gamma_2)^c$ such that $N_i = N_{i1} \cap N_{i2} \forall i \in I$ which implies that:

$$\bigcap_{i \in I} N_i = \bigcap_{i \in I} (N_{i1} \cap N_{i2}) = (\bigcap_{i \in I} [N_{i1}]) \cap (\bigcap_{i \in I} [N_{i2}]).$$

Now, since Γ_1, Γ_2 are neutrosophic crisp supra topologies, $\bigcap_{i \in I} [N_{i1}] \in (\Gamma_1)^c$ and $\bigcap_{i \in I} [N_{i2}] \in (\Gamma_2)^c$. Therefore,

$\bigcap_{i \in I} N_i$ is a pairwise neutrosophic crisp supra closed set.

Remark 5.6.

- 1) Every neutrosophic crisp supra open sets in (X, Γ_1) or (X, Γ_2) is a pairwise neutrosophic crisp supra open set.
- 2) Every neutrosophic crisp supra closed sets in (X, Γ_1) or (X, Γ_2) is a pairwise neutrosophic crisp supra closed set.

Proof. Straightforward.

Remark 5.7.

Let (X, Γ_1, Γ_2) be a neutrosophic crisp supra bi-topological space then :

- 1) Every neutrosophic crisp supra bi-open sets is a pairwise neutrosophic crisp supra open set, but the converse is not true.
- 2) Every neutrosophic crisp supra bi-closed sets is a pairwise neutrosophic crisp supra closed set, but the converse is not true.

Proof. Straightforward.

Example 5.8.

In Example 3.9 $A \cup B$ is pairwise neutrosophic crisp supra open sets in (X, Γ_1, Γ_2) , but it is not a neutrosophic crisp supra bi-open set .

Definition 5.9.

Let (X, Γ_1, Γ_2) be a neutrosophic crisp supra bi-topological space (SBI-NCTS) and let A be a neutrosophic crisp set. then the union of any neutrosophic crisp a pairwise supra open sets , contained in A is called pairwise neutrosophic crisp supra interior of A ($PNS^{Bi}int(A)$). and let A be a neutrosophic crisp set.

$$PNS^{Bi}int(A) = \cup \{B : B \subseteq A ; B \in PNC SO(X, \Gamma_1, \Gamma_2)\}.$$

Theorem 5.10.

Let (X, Γ_1, Γ_2) be a neutrosophic crisp supra bi-topological space, and let A be a neutrosophic crisp set. then :

1. $PNS^{Bi}int(A) \subseteq A$.
2. $PNS^{Bi}int(A)$ is pairwise neutrosophic crisp supra open set .

Proof :

1. The proof follows from the definition of $PNS^{Bi}int(A)$ as a union of any pairwise neutrosophic crisp supra open sets ,contained in A .
2. The proof follows from Theorem 5.4.

Definition 5.11.

Let (X, Γ_1, Γ_2) be neutrosophic crisp supra bi-topological space (SBI-NCTS) and let A be a neutrosophic crisp set. Then the intersection of any neutrosophic crisp a pairwise supra closed sets , containing A is called pairwise neutrosophic crisp supra closer of A ($PNS^{Bi}cl(A)$).

$$PNS^{Bi}cl(A) = \cup \{B : B \supseteq A ; B \in PNC SC(X, \Gamma_1, \Gamma_2)\}.$$

Theorem 5.12.

Let (X, Γ_1, Γ_2) be neutrosophic crisp supra bi-topological space, A be a neutrosophic crisp set then :

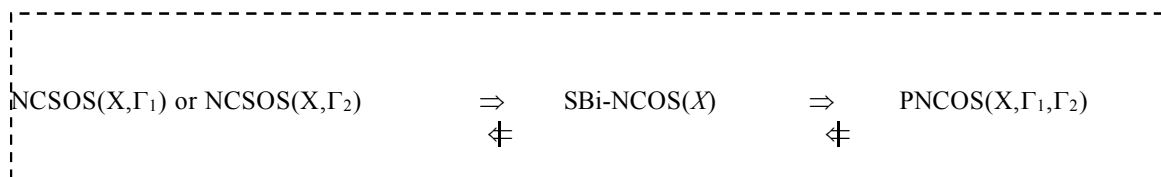
1. $A \subseteq \text{PNS}^{\text{Bi cl}}(A)$.
2. $\text{PNS}^{\text{Bi cl}}(A)$ is a pairwise neutrosophic crisp supra closed set.

Proof :

1. The proof follows from the definition of $\text{PNS}^{\text{Bi cl}}(A)$ as a intersection of any pairwise neutrosophic crisp supra closed set containing in A .
2. The proof follows from remark 5.5.

Remark 5.13.

The following diagram shows the relationship between different types of neutrosophic crisp open sets that were studied in section 3 and section 5.

**6. Conclusion**

In this paper, we have defined a new topological space by using neutrosophic crisp sets due to Salama [3]. This new space called neutrosophic crisp supra bi-topological space. Then we have introduced new neutrosophic crisp open(closed) sets in neutrosophic crisp supra bi-topological space. Also we studied some of their basic properties and their relationship with each other. We introduced pairwise neutrosophic crisp supra closure, pairwise neutrosophic crisp supra interior, we also have provided examples where such properties fail to be preserved. In addition, Many results have been established. This paper is just the beginning of a new structure, and we have studied a few ideas only, it will be necessary to carry out more theoretical research to establish a general framework for the practical application. In the future, using these notions, various classes of mappings on neutrosophic crisp supra bi-topological space, separation axioms on the neutrosophic crisp supra bi-topological spaces, neutrosophic crisp supra bi- α -open sets, neutrosophic crisp supra bi- β -open sets, neutrosophic crisp supra bi-pre-open sets, Neutrosophic crisp supra bi-semi-open sets and many researchers can be studied.

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Exponential Laws and Aggregation Operators on Neutrosophic Cubic Sets

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Abstract

This paper presents operational laws along with their cosine measure for the numbers whose base is an interval value and study their properties. Consequent upon these definitions and properties neutrosophic cubic weighted exponential averaging and dual neutrosophic cubic weighted exponential averaging aggregation operators are defined. A multi attribute decision making method is then developed for proposed aggregation operators. An example is constructed as an application. The validity of multi attribute decision making method is also tested and comparative analysis is provided to compare these aggregation operators with existing results.

Keywords: : Neutrosophic cubic number; dual neutrosophic cubic number; neutrosophic cubic exponential weighted averaging; dual neutrosophic cubic exponential weighted averaging ; multi attribute decision making.

1.Introduction

The decision making is an imperative part of cognitive based human activity. Multi attribute decision making method (MADM) provides a better environment to rank the set of alternatives under different criteria. The problem arises when vague and insufficient data is available. This uncertainty in data can be handle by fuzzy set theory (FS) presented by Zadeh [1] and its extensions like interval valued fuzzy set (IVFS) [2,3], intuitionistic fuzzy set [4], interval valued intuitionistic fuzzy set (IVIFS) [5], cubic set (CS) [6]. The intuitionistic fuzzy set attracted the researcher due to its structure of both membership, non-membership and hesitant component. Over the last few decades, several researchers used it for decision making problems [7,8,9,10,11,12]. In IFS the hesitancy component is depended upon the choice of membership degree and non-membership degree which restricted the choice of choosing. Smarandache defined a novel tool, neutrosophic set NS [13] to deal with vagueness in more desirable way. NS are generalization of IFS [14]. In NS all the components are independent. Soon after its presentation, it is further extended into INS [15], NCS [16] etc. The collection and manipulation of data is a hard job. In daily life problems we are often in a situation that the extraction of accurate and precise information is not possible due to the vague nature of problem, communication gaps, hesitancy etc. Thus NS and its extension are better tools to deal with such situation in comparison with IFS and cubic set. This characteristic of NS and INS attracted the researchers to apply it in different field of decision making process [17,18,19,20,21,22,23,24,25]. Neutrosophic Cubic Number (NCN) is a combination of INS and NS which makes it a better choice so that expert can choose the value in the form of an interval value and single value. NCS provides a better plate form to deal with vague and insufficient data. In recent pass researchers have

applied NCS to aggregation operators and decision making problem. Zhan *et al.* [26] worked on multi criteria decision making on neutrosophic cubic sets. Banerjee *et al.* [27] used GRA for multi criteria decision making on neutrosophic cubic set. Lu and Ye [28] defined cosine measure to neutrosophic cubic set. Pramanik *et al.* [29] used similarity measure to neutrosophic cubic set in 2017. Majid *et al.* [30] presented neutrosophic cubic Einstein geometric aggregation operators. Khalid *et al.* [31] introduced MBJ – Neutrosophic Translation on G-Algebra and Schweizer [32] studied two probabilities for the three states of neutrosophy. In their works all these researchers defined arithmetic and Einstein operation and aggregations operators under neutrosophic cubic environment. It is observed in decision making process that the weight is based on the assumption to be in $[0,1]$ and their sum is one. The aim of this work is to deal the situation where the weight is crisp or interval value. To support this task some new operational laws along with some aggregation operators are defined on NCS.

Contribution In this work, we propose methodologies to measures the aggregate value of neutrosophic cubic values using exponential form.

- The neutrosophic cubic exponential operational laws are defined along with some important properties.
- Based on these operations and properties neutrosophic cubic exponential aggregations operators are defined. Validity and comparative analysis are discussed.
- The neutrosophic cubic set is the combination of both neutrosophic and interval neutrosophic, this characteristic enable us to deal both interval valued neutrosophic and neutrosophic set at the same time.

Organization This study consist of nine sections. Section 1 covers the introductory work of researchers over the some past decades. In section 2 the preliminaries work is reviewed which enable us to start this work. In section 3 some exponential operational laws with base crisp and interval value are introduced, dual neutrosophic cubic number (DNCN) are defined and some useful properties are obtained. Cosine measures is defined to compare two NCN and DNCN. Section 4, presents neutrosophic cubic weighted exponential averaging (NCWEA) operators and dual neutrosophic cubic weighted exponential averaging (DNCWEA) aggregations operators are defined in which the weight is in the form of crisp value or interval valued and the exponents are NCS. Section 5 develops a decision making process for the proposed aggregations operators. An illustrative example is provided as an application in section 6. In Section 7, the validity test is performed for DM problem to check the validity of MADM. In section 8, the MADM based upon proposed aggregation operators is compared with some neutrosophic cubic weighted averaging (NCWA) and neutrosophic cubic Einstein weighted averaging (NCEWA) aggregation operators. The paper ends with conclusion in section 9.

2. Preliminaries

Definition 2.1 [13] A structure $N = \{(T_N(y), I_N(y), F_N(y)) | y \in Y\}$ NS, where T_N , I_N and F_N are fuzzy sets and respectively called truth, indeterminacy and falsity functions.

Definition 2.2 [15] An INS in Y is a structure $N = \{(\tilde{T}_N(y), \tilde{I}_N(y), \tilde{F}_N(y)) | y \in Y\}$ where $\tilde{T}_N(y)$, $\tilde{I}_N(y)$ and $\tilde{F}_N(y)$ are interval valued fuzzy truth, indeterminacy an falsity function in Y respectively.

Definition 2.3 [16] A structure $A = \{(y, \tilde{T}_N(y), \tilde{I}_N(y), \tilde{F}_N(y), T_N(y), I_N(y), F_N(y)) / y \in Y\}$ is NCS in Y in which $(\tilde{T}_N(y) = [T_N^L, T_N^U], \tilde{I}_N(y) = [I_N^L, I_N^U], \tilde{F}_N(y) = [F_N^L, F_N^U])$ is an INS and (T_N, I_N, F_N) is NS in Y . Simply denoted by $N = (\tilde{T}_N, \tilde{I}_N, \tilde{F}_N, T_N, I_N, F_N)$ where, $[0, 0] \leq \tilde{T}_N + \tilde{I}_N + \tilde{F}_N \leq [3, 3]$, $0 \leq T_N + I_N + F_N \leq 3$. N^Y denotes the collection of NCS in Y .

For the sake of convenience the NCS are written as $A = ([T_A^L, T_A^U], [I_A^L, I_A^U], [F_A^L, F_A^U], T_A, I_A, F_A)$

Definition 2.4 [30] The sum of two NCN, $A = ([T_A^L, T_A^U], [I_A^L, I_A^U], [F_A^L, F_A^U], T_A, I_A, F_A)$ and $B = ([T_B^L, T_B^U], [I_B^L, I_B^U], [F_B^L, F_B^U], T_B, I_B, F_B)$ is defined as

$$A \oplus B = \left([T_A^L + T_B^L - T_A^L T_B^L, T_A^U + T_B^U - T_A^U T_B^U], [I_A^L + I_B^L - I_A^L I_B^L, I_A^U + I_B^U - I_A^U I_B^U], [F_A^L F_B^L, F_A^U F_B^U], T_A T_B, I_A I_B, F_A + F_B - F_A F_B \right)$$

Definition 2.5 [30] The product of two NCN, $A = ([T_A^L, T_A^U], [I_A^L, I_A^U], [F_A^L, F_A^U], T_A, I_A, F_A)$ and $B = ([T_B^L, T_B^U], [I_B^L, I_B^U], [F_B^L, F_B^U], T_B, I_B, F_B)$ is defined as

$$A \otimes B = \left([T_A^L T_B^L, T_A^U T_B^U], [I_A^L I_B^L, I_A^U I_B^U], [F_A^L + F_B^L - F_A^L F_B^L, F_A^U + F_B^U - F_A^U F_B^U], T_A + T_B - T_A T_B, I_A + I_B - I_A I_B, F_A F_B \right)$$

Definition 2.6 [30] The scalar multiplication on a NCN $A = ([T_A^L, T_A^U], [I_A^L, I_A^U], [F_A^L, F_A^U], T_A, I_A, F_A)$ and a Scalar ϖ is defined

$$\varpi A = \left([1 - (1 - T_A^L)^\varpi, 1 - (1 - T_A^U)^\varpi], [1 - (1 - I_A^L)^\varpi, 1 - (1 - I_A^U)^\varpi], [(F_A^L)^\varpi, (F_A^U)^\varpi], (T_A)^\varpi, (I_A)^\varpi, 1 - (1 - F_A)^\varpi \right)$$

Theorem 2.7 [30] Let $A = ([T_A^L, T_A^U], [I_A^L, I_A^U], [F_A^L, F_A^U], T_A, I_A, F_A)$ be a NCN, then the exponential operation is defined by

$$A^\varpi = \left([(T_A^L)^\varpi, (T_A^U)^\varpi], [(I_A^L)^\varpi, (I_A^U)^\varpi], [1 - (1 - F_A^L)^\varpi, 1 - (1 - F_A^U)^\varpi], 1 - (1 - T_A)^\varpi, 1 - (1 - I_A)^\varpi, (F_A)^\varpi \right)$$

3. Exponential operational laws with crisp and interval parameters on neutrosophic cubic sets

Operational laws has a key role in any work. In this section we develop some exponential laws in neutrosophic cubic environment in which the exponential parameters are crisp and interval value.

Definition 3.1 Let $A = ([T_A^L, T_A^U], [I_A^L, I_A^U], [F_A^L, F_A^U], T_A, I_A, F_A)$ be a NCN the exponential law for crisp value

∇ is defined as

$$\nabla^A = \begin{cases} \left(\begin{aligned} &\left[\nabla^{1-(T_A)^L}, \nabla^{1-(T_A)^U} \right], \left[\nabla^{1-(I_A)^L}, \nabla^{1-(I_A)^U} \right], \\ &\left[1 - \nabla^{(F_A)^L}, 1 - \nabla^{(F_A)^U} \right], 1 - \nabla^{T_A}, 1 - \nabla^{I_A}, \nabla^{1-F_A} \end{aligned} \right), \nabla \in [0, 1] \\ \left(\begin{aligned} &\left[(1/\nabla)^{1-(T_A)^L}, (1/\nabla)^{1-(T_A)^U} \right], \left[(1/\nabla)^{1-(I_A)^L}, (1/\nabla)^{1-(I_A)^U} \right], \\ &\left[1 - (1/\nabla)^{(F_A)^L}, 1 - (1/\nabla)^{(F_A)^U} \right], 1 - (1/\nabla)^{T_A}, 1 - (1/\nabla)^{I_A}, (1/\nabla)^{1-F_A} \end{aligned} \right), \nabla > 1 \end{cases}$$

In both cases ∇^A is a NCN.

Example 3.2 Let $A = ([0.2, 0.8], [0.4, 0.7], [0.1, 0.5], 0.7, 0.2, 0.6)$ be a NCN, $\nabla = 0.5$ and $\nabla = 3$, then

$$\nabla^A = \begin{cases} ([0.574, 0.870], [0.659, 0.812], [0.066, 0.292], 0.384, 0.129, 0.757), \nabla = 0.5 \\ ([0.801, 0.411], [0.641, 0.460], [0.104, 0.425], 0.539, 0.198, 0.514), \nabla = 3 \end{cases}$$

Definition 3.3 Let $A = ([T_A^L, T_A^U], [I_A^L, I_A^U], [F_A^L, F_A^U], T_A, I_A, F_A)$, be a NCN and

$A^* = ([1, 1], [1, 1], [0, 0], 0, 0, 1)$ be maximum NCN, then the cosine measure(C_m) is defined as

$$C_m(A) = \left\{ \cos \frac{\pi}{18} (1 - T_A^L + 1 - T_A^U + 1 - I_A^L + 1 - I_A^U + F_A^L + F_A^U + T_A + I_A + 1 - F_A) \right\}, C_m(A) \in [0, 1]$$

Remark 3.4 If $C_m(A)$ and $C_m(B)$ be the cosine measures of two NCN then $C_m(A) > C_m(B) \Rightarrow A > B$ and

$$C_m(A) = C_m(B) \Rightarrow A = B$$

Theorem 3.5 Let $A = ([\nabla^{1-T_A^L}, \nabla^{1-T_A^U}], [\nabla^{1-I_A^L}, \nabla^{1-I_A^U}], [1 - \nabla^{F_A^L}, 1 - \nabla^{F_A^U}], 1 - \nabla^{T_A}, 1 - \nabla^{I_A}, \nabla^{1-F_A})$ be a neutrosophic cubic value and $\nabla_1 \geq \nabla_2$, then $(\nabla_1)^A \geq (\nabla_2)^A$ for $\nabla_1, \nabla_2 \in [0, 1]$ and $(\nabla_1)^A \leq (\nabla_2)^A$ for $\nabla_1, \nabla_2 > 1$.

Proof: Let $\nabla_1 \geq \nabla_2$ and $\nabla_1, \nabla_2 \in [0, 1]$ then

$$(\nabla_1)^A = \left(\begin{aligned} &\left[(\nabla_1)^{1-T_A^L}, (\nabla_1)^{1-T_A^U} \right], \left[(\nabla_1)^{1-I_A^L}, (\nabla_1)^{1-I_A^U} \right], \\ &\left[1 - (\nabla_1)^{F_A^L}, 1 - (\nabla_1)^{F_A^U} \right], 1 - (\nabla_1)^{T_A}, 1 - (\nabla_1)^{I_A}, (\nabla_1)^{1-F_A} \end{aligned} \right)$$

and

$$(\nabla_2)^A = \left(\begin{aligned} &\left[(\nabla_2)^{1-T_A^L}, (\nabla_2)^{1-T_A^U} \right], \left[(\nabla_2)^{1-I_A^L}, (\nabla_2)^{1-I_A^U} \right], \\ &\left[1 - (\nabla_2)^{F_A^L}, 1 - (\nabla_2)^{F_A^U} \right], 1 - (\nabla_2)^{T_A}, 1 - (\nabla_2)^{I_A}, (\nabla_2)^{1-F_A} \end{aligned} \right)$$

Since $(\nabla_1)^{1-T_A^L} \geq (\nabla_2)^{1-T_A^L}, (\nabla_1)^{1-T_A^U} \geq (\nabla_2)^{1-T_A^U}, (\nabla_1)^{1-I_A^L} \geq (\nabla_2)^{1-I_A^L}, (\nabla_1)^{1-I_A^U} \geq (\nabla_2)^{1-I_A^U},$

$$1 - (\nabla_1)^{F_A^L} \leq 1 - (\nabla_2)^{F_A^L}, 1 - (\nabla_1)^{F_A^U} \leq 1 - (\nabla_2)^{F_A^U}, 1 - (\nabla_1)^{T_A} \leq 1 - (\nabla_2)^{T_A},$$

$$1 - (\nabla_1)^{I_A} \leq 1 - (\nabla_2)^{I_A}, (\nabla_1)^{1-F_A} \geq (\nabla_2)^{1-F_A}, \text{ so}$$

$$\cos((\nabla_1)^A) = \left\{ \frac{\pi}{18} \left(\frac{1 - (\nabla_1)^{1-T_A^L} + 1 - (\nabla_1)^{1-I_A^L} + (\nabla_1)^{F_A^L} + (\nabla_1)^{F_A^U} + (\nabla_1)^{T_A^U} + (\nabla_1)^{I_A^U} + 1 - (\nabla_1)^{1-F_A}}{1 - (\nabla_1)^{1-T_A^U} + 1 - (\nabla_1)^{1-I_A^U} + (\nabla_1)^{F_A^U} + (\nabla_1)^{F_A^L} + (\nabla_1)^{T_A^L} + (\nabla_1)^{I_A^L} + 1 - (\nabla_1)^{1-F_A}} \right) \right\} \geq$$

$$\cos((\nabla_2)^A) = \left\{ \frac{\pi}{18} \left(\frac{1 - (\nabla_2)^{1-T_A^L} + 1 - (\nabla_2)^{1-I_A^L} + (\nabla_2)^{F_A^L} + (\nabla_2)^{F_A^U} + (\nabla_2)^{T_A^U} + (\nabla_2)^{I_A^U} + 1 - (\nabla_2)^{1-F_A}}{1 - (\nabla_2)^{1-T_A^U} + 1 - (\nabla_2)^{1-I_A^U} + (\nabla_2)^{F_A^U} + (\nabla_2)^{F_A^L} + (\nabla_2)^{T_A^L} + (\nabla_2)^{I_A^L} + 1 - (\nabla_2)^{1-F_A}} \right) \right\}$$

obviously $(\nabla_1)^A \geq (\nabla_2)^A$, if $\nabla_1 \geq \nabla_2$ and $\nabla_1, \nabla_2 > 1$, then $0 < 1/\nabla_1, 1/\nabla_2 \leq 1$.

Remark 3.6 Considering some values of ∇ , we can affirm some special cases of $(\nabla)^A$.

1. If $\nabla = 1$, then $(\nabla)^A = ([1, 1], [1, 1], [0, 0], 0, 0, 1)$.
2. If $A = ([1, 1], [1, 1], [0, 0], 0, 0, 1)$, then $(\nabla)^A = ([1, 1], [1, 1], [0, 0], 0, 0, 1)$. For each value of ∇ .
3. If $A = ([0, 0], [0, 0], [1, 1], 1, 1, 0)$, then $(\nabla)^A = ([\nabla, \nabla], [\nabla, \nabla], [1 - (\nabla)^{F_A^L}, 1 - (\nabla)^{F_A^U}], 1 - (\nabla)^{T_A}, 1 - (\nabla)^{I_A}, \nabla)$.

Theorem 3.7 Let $A = ([T_A^L, T_A^U], [I_A^L, I_A^U], [F_A^L, F_A^U], T_A, I_A, F_A)$, $B = ([T_B^L, T_B^U], [I_B^L, I_B^U], [F_B^L, F_B^U], T_B, I_B, F_B)$ and $C = ([T_C^L, T_C^U], [I_C^L, I_C^U], [F_C^L, F_C^U], T_C, I_C, F_C)$ be three NCNs and $\nabla \in [0, 1]$ (if $\nabla > 1$, then $1/\nabla$) then the following holds.

- $\nabla^A \oplus \nabla^B = \nabla^B \oplus \nabla^A$
- $\nabla^A \otimes \nabla^B = \nabla^B \otimes \nabla^A$
- $(\nabla^A \oplus \nabla^B) \oplus \nabla^C = \nabla^A \oplus (\nabla^B \oplus \nabla^C)$
- $\nabla^A \otimes \nabla^B \otimes \nabla^C = \nabla^A \otimes (\nabla^B \otimes \nabla^C)$

Proof Straight forward, so omitted.

Theorem 3.8 Let $A = ([T_A^L, T_A^U], [I_A^L, I_A^U], [F_A^L, F_A^U], T_A, I_A, F_A)$, $B = ([T_B^L, T_B^U], [I_B^L, I_B^U], [F_B^L, F_B^U], T_B, I_B, F_B)$ be two NCNs and ϖ be a scalar then the following holds.

- $\varpi(\nabla^A \oplus \nabla^B) = \varpi \nabla^A \oplus \varpi \nabla^B$
- $(\nabla^A \oplus \nabla^B)^\varpi = (\nabla^A)^\varpi \oplus (\nabla^B)^\varpi$
- $(\varpi_1 \oplus \varpi_2) \nabla^A = \varpi_1 \nabla^A \oplus \varpi_2 \nabla^A$
- $((\nabla^A)^\varpi_1)^\varpi_2 = (\nabla^A)^{\varpi_1 \varpi_2}$

Proof: Consider

$$\begin{aligned}
\varpi(\nabla^A \oplus \nabla^B) &= \varpi \left\{ \left(\begin{bmatrix} \nabla^{1-T_A^L}, \nabla^{1-T_A^U} \\ \nabla^{1-I_A^L}, \nabla^{1-I_A^U} \end{bmatrix}, \begin{bmatrix} 1 - \nabla^{F_A^L}, 1 - \nabla^{F_A^U} \\ 1 - \nabla^{T_A}, 1 - \nabla^{I_A}, \nabla^{1-F_A} \end{bmatrix} \right) \oplus \left(\begin{bmatrix} \nabla^{1-T_B^L}, \nabla^{1-T_B^U} \\ \nabla^{1-I_B^L}, \nabla^{1-I_B^U} \end{bmatrix}, \begin{bmatrix} 1 - \nabla^{F_B^L}, 1 - \nabla^{F_B^U} \\ 1 - \nabla^{T_B}, 1 - \nabla^{I_B}, \nabla^{1-F_B} \end{bmatrix} \right) \right\} \\
&= \varpi \left\{ \left(\begin{bmatrix} \nabla^{1-T_A^L} + \nabla^{1-T_B^L} - \nabla^{1-T_A^L} \nabla^{1-T_B^L}, \nabla^{1-T_A^U} + \nabla^{1-T_B^U} - \nabla^{1-T_A^U} \nabla^{1-T_B^U} \\ \nabla^{1-I_A^L} + \nabla^{1-I_B^L} - \nabla^{1-I_A^L} \nabla^{1-I_B^L}, \nabla^{1-I_A^U} + \nabla^{1-I_B^U} - \nabla^{1-I_A^U} \nabla^{1-I_B^U} \\ \left[(1 - \nabla^{F_A^L})(1 - \nabla^{F_B^L}), (1 - \nabla^{F_A^U})(1 - \nabla^{F_B^U}) \right] \\ (1 - \nabla^{T_A})(1 - \nabla^{T_B}), (1 - \nabla^{I_A})(1 - \nabla^{I_B}), \nabla^{1-F_A} + \nabla^{1-F_B} - \nabla^{1-F_A} \nabla^{1-F_B} \end{bmatrix} \right) \right\} \\
&= \left\{ \begin{bmatrix} 1 - (1 - \nabla^{1-T_A^L} - \nabla^{1-T_B^L} \nabla^{1-T_A^L} \nabla^{1-T_B^L})^{\varpi} \\ 1 - (1 - \nabla^{1-T_A^U} - \nabla^{1-T_B^U} \nabla^{1-T_A^U} \nabla^{1-T_B^U})^{\varpi} \end{bmatrix}, \begin{bmatrix} 1 - (1 - \nabla^{1-I_A^L} - \nabla^{1-I_B^L} \nabla^{1-I_A^L} \nabla^{1-I_B^L})^{\varpi} \\ 1 - (1 - \nabla^{1-I_A^U} - \nabla^{1-I_B^U} \nabla^{1-I_A^U} \nabla^{1-I_B^U})^{\varpi} \end{bmatrix}, \right. \\
&\quad \left. \begin{bmatrix} \left[(1 - \nabla^{F_A^L})(1 - \nabla^{F_B^L}) \right]^{\varpi}, \left[(1 - \nabla^{F_A^U})(1 - \nabla^{F_B^U}) \right]^{\varpi} \\ (1 - \nabla^{T_A})(1 - \nabla^{T_B})^{\varpi}, (1 - \nabla^{I_A})(1 - \nabla^{I_B})^{\varpi} \\ (1 - (1 - \nabla^{1-F_A} \nabla^{1-F_B} + \nabla^{1-F_A} \nabla^{1-F_B})^{\varpi}) \end{bmatrix} \right\} \\
&= \left\{ \begin{bmatrix} 1 - (1 - \nabla^{1-T_A^L})^{\varpi} (1 - \nabla^{1-T_B^L})^{\varpi} \\ 1 - (1 - \nabla^{1-T_A^U})^{\varpi} (1 - \nabla^{1-T_B^U})^{\varpi} \end{bmatrix}, \begin{bmatrix} 1 - (1 - \nabla^{1-I_A^L})^{\varpi} (1 - \nabla^{1-I_B^L})^{\varpi} \\ 1 - (1 - \nabla^{1-I_A^U})^{\varpi} (1 - \nabla^{1-I_B^U})^{\varpi} \end{bmatrix}, \right. \\
&\quad \left. \begin{bmatrix} (1 - \nabla^{F_A^L})^{\varpi} (1 - \nabla^{F_B^L})^{\varpi}, (1 - \nabla^{F_A^U})^{\varpi} (1 - \nabla^{F_B^U})^{\varpi} \\ (1 - \nabla^{T_A})^{\varpi} (1 - \nabla^{T_B})^{\varpi}, (1 - \nabla^{I_A})^{\varpi} (1 - \nabla^{I_B})^{\varpi} \\ 1 - (1 - \nabla^{1-F_A})^{\varpi} (1 - \nabla^{1-F_B})^{\varpi} \end{bmatrix} \right\}
\end{aligned}$$

$$\begin{aligned}
&= \left\{ \begin{aligned} &\begin{bmatrix} 1 - \left(1 - \nabla^{1-T_A^L}\right)^{\overline{\sigma}} - \left(1 - \nabla^{1-T_B^L}\right)^{\overline{\sigma}} + \left(1 - \nabla^{1-T_A^L}\right)^{\overline{\sigma}} + \\ &\quad \left(1 - \nabla^{1-T_B^L}\right)^{\overline{\sigma}} - \left(1 - \nabla^{1-T_A^L}\right)^{\overline{\sigma}} \left(1 - \nabla^{1-T_B^L}\right)^{\overline{\sigma}}, \\ &1 - \left(1 - \nabla^{1-T_A^U}\right)^{\overline{\sigma}} - \left(1 - \nabla^{1-T_B^U}\right)^{\overline{\sigma}} + \left(1 - \nabla^{1-T_A^U}\right)^{\overline{\sigma}} + \\ &\quad \left(1 - \nabla^{1-T_B^U}\right)^{\overline{\sigma}} - \left(1 - \nabla^{1-T_A^U}\right)^{\overline{\sigma}} \left(1 - \nabla^{1-T_B^U}\right)^{\overline{\sigma}} \end{bmatrix}, \\ &\begin{bmatrix} 1 - \left(1 - \nabla^{1-I_A^L}\right)^{\overline{\sigma}} - \left(1 - \nabla^{1-I_B^L}\right)^{\overline{\sigma}} + \left(1 - \nabla^{1-I_A^L}\right)^{\overline{\sigma}} + \\ &\quad \left(1 - \nabla^{1-I_B^L}\right)^{\overline{\sigma}} - \left(1 - \nabla^{1-I_A^L}\right)^{\overline{\sigma}} \left(1 - \nabla^{1-I_B^L}\right)^{\overline{\sigma}}, \\ &1 - \left(1 - \nabla^{1-I_A^U}\right)^{\overline{\sigma}} - \left(1 - \nabla^{1-I_B^U}\right)^{\overline{\sigma}} + \left(1 - \nabla^{1-I_A^U}\right)^{\overline{\sigma}} + \\ &\quad \left(1 - \nabla^{1-I_B^U}\right)^{\overline{\sigma}} - \left(1 - \nabla^{1-I_A^U}\right)^{\overline{\sigma}} \left(1 - \nabla^{1-I_B^U}\right)^{\overline{\sigma}} \end{bmatrix}, \\ &\left[\varpi \nabla^{F_A^L} \oplus \varpi \nabla^{F_B^L}, \varpi \nabla^{F_A^U} \oplus \varpi \nabla^{F_B^U} \right], \varpi \nabla^{T_A} \oplus \varpi \nabla^{T_B}, \varpi \nabla^{I_A} \oplus \varpi \nabla^{I_B}, \\ &\quad 1 - \left(1 - \nabla^{1-T_A}\right)^{\overline{\sigma}} - \left(1 - \nabla^{1-T_B}\right)^{\overline{\sigma}} + \left(1 - \nabla^{1-T_A}\right)^{\overline{\sigma}} + \\ &\quad \left(1 - \nabla^{1-T_B}\right)^{\overline{\sigma}} - \left(1 - \nabla^{1-T_A}\right)^{\overline{\sigma}} \left(1 - \nabla^{1-T_B}\right)^{\overline{\sigma}} \end{aligned} \right\} \\
&= \left\{ \begin{aligned} &\begin{bmatrix} 1 - \left(1 - \nabla^{1-T_A^L}\right)^{\overline{\sigma}} + 1 - \left(1 - \nabla^{1-T_B^L}\right)^{\overline{\sigma}} - \left(1 - \left(1 - \nabla^{1-T_A^L}\right)^{\overline{\sigma}} \left(1 - \nabla^{1-T_B^L}\right)^{\overline{\sigma}} \right), \\ &1 - \left(1 - \nabla^{1-T_A^U}\right)^{\overline{\sigma}} + 1 - \left(1 - \nabla^{1-T_B^U}\right)^{\overline{\sigma}} - \left(1 - \left(1 - \nabla^{1-T_A^U}\right)^{\overline{\sigma}} \left(1 - \nabla^{1-T_B^U}\right)^{\overline{\sigma}} \right) \end{bmatrix}, \\ &\begin{bmatrix} 1 - \left(1 - \nabla^{1-I_A^L}\right)^{\overline{\sigma}} + 1 - \left(1 - \nabla^{1-I_B^L}\right)^{\overline{\sigma}} - \left(1 - \left(1 - \nabla^{1-I_A^L}\right)^{\overline{\sigma}} \left(1 - \nabla^{1-I_B^L}\right)^{\overline{\sigma}} \right), \\ &1 - \left(1 - \nabla^{1-I_A^U}\right)^{\overline{\sigma}} + 1 - \left(1 - \nabla^{1-I_B^U}\right)^{\overline{\sigma}} - \left(1 - \left(1 - \nabla^{1-I_A^U}\right)^{\overline{\sigma}} \left(1 - \nabla^{1-I_B^U}\right)^{\overline{\sigma}} \right) \end{bmatrix}, \\ &\left[\varpi \nabla^{F_A^L} \oplus \varpi \nabla^{F_B^L}, \varpi \nabla^{F_A^U} \oplus \varpi \nabla^{F_B^U} \right], \varpi \nabla^{T_A} \oplus \varpi \nabla^{T_B}, \varpi \nabla^{I_A} \oplus \varpi \nabla^{I_B} \end{aligned} \right\}
\end{aligned}$$

$$\begin{aligned}
&= \left\{ \begin{aligned} &\left[\left(1 - \left(1 - \nabla^{1-T_A^L} \right)^{\varpi} \right) \oplus \left(1 - \left(1 - \nabla^{1-T_B^L} \right)^{\varpi} \right), \right. \\ &\left. \left[\left(1 - \left(1 - \nabla^{1-T_A^U} \right)^{\varpi} \right) \oplus \left(1 - \left(1 - \nabla^{1-T_B^U} \right)^{\varpi} \right) \right] \right. \\ &\left[\left(1 - \left(1 - \nabla^{1-I_A^L} \right)^{\varpi} \right) \oplus \left(1 - \left(1 - \nabla^{1-I_B^L} \right)^{\varpi} \right), \right. \\ &\left. \left[\left(1 - \left(1 - \nabla^{1-I_A^U} \right)^{\varpi} \right) \oplus \left(1 - \left(1 - \nabla^{1-I_B^U} \right)^{\varpi} \right) \right] \right. \\ &\left[\varpi \nabla^{F_A^L} \oplus \varpi \nabla^{F_B^L}, \varpi \nabla^{F_A^U} \oplus \varpi \nabla^{F_B^U} \right], \varpi \nabla^{T_A} \oplus \varpi \nabla^{T_B}, \varpi \nabla^{I_A} \oplus \varpi \nabla^{I_B}, \\ &1 - \left(1 - \nabla^{1-T_A} \right)^{\varpi} + 1 - \left(1 - \nabla^{1-T_B} \right)^{\varpi} - \left(1 - \left(1 - \nabla^{1-T_A} \right)^{\varpi} \left(1 - \nabla^{1-T_B} \right)^{\varpi} \right) \end{aligned} \right\} \\
&= \left\{ \begin{aligned} &\left[\varpi \nabla^{T_A^L} \oplus \varpi \nabla^{T_B^L}, \varpi \nabla^{T_A^U} \oplus \varpi \nabla^{T_B^U} \right], \left[\varpi \nabla^{I_A^L} \oplus \varpi \nabla^{I_B^L}, \varpi \nabla^{I_A^U} \oplus \varpi \nabla^{I_B^U} \right], \\ &\left[\varpi \nabla^{F_A^L} \oplus \varpi \nabla^{F_B^L}, \varpi \nabla^{F_A^U} \oplus \varpi \nabla^{F_B^U} \right], \varpi \nabla^{T_A} \oplus \varpi \nabla^{T_B}, \varpi \nabla^{I_A} \oplus \varpi \nabla^{I_B}, \varpi \nabla^{F_A} \oplus \varpi \nabla^{F_B} \end{aligned} \right\} \\
&= \varpi \nabla^A \oplus \varpi \nabla^B \\
\triangleright \quad (\varpi_1 + \varpi_2) \nabla^A &= (\varpi_1 + \varpi_2) \left\{ \begin{aligned} &\left[\nabla^{1-T_A^L}, \nabla^{1-T_A^U} \right], \left[\nabla^{1-I_A^L}, \nabla^{1-I_A^U} \right], \\ &\left[1 - \nabla^{F_A^L}, 1 - \nabla^{F_A^U} \right], 1 - \nabla^{T_A}, 1 - \nabla^{I_A}, \nabla^{1-F_A} \end{aligned} \right\} \\
&= \left\{ \begin{aligned} &\left[1 - \left(1 - \nabla^{1-T_A^L} \right)^{(\varpi_1 + \varpi_2)}, 1 - \left(1 - \nabla^{1-T_A^U} \right)^{(\varpi_1 + \varpi_2)} \right], \\ &\left[1 - \left(1 - \nabla^{1-I_A^L} \right)^{(\varpi_1 + \varpi_2)}, 1 - \left(1 - \nabla^{1-I_A^U} \right)^{(\varpi_1 + \varpi_2)} \right], \\ &\left[\left(1 - \nabla^{F_A^L} \right)^{(\varpi_1 + \varpi_2)}, \left(1 - \nabla^{F_A^U} \right)^{(\varpi_1 + \varpi_2)} \right], \\ &\left(1 - \nabla^{T_A} \right)^{(\varpi_1 + \varpi_2)}, \left(1 - \nabla^{I_A} \right)^{(\varpi_1 + \varpi_2)}, 1 - \left(1 - \nabla^{1-F_A} \right)^{(\varpi_1 + \varpi_2)} \end{aligned} \right\}
\end{aligned}$$

$$\begin{aligned}
&= \left\{ \left[\begin{aligned} &1 - \left(1 - \nabla^{1-T_A^L}\right)^{\overline{\omega}_1} + \left(1 - \nabla^{1-T_A^L}\right)^{\overline{\omega}_1} - \left(1 - \nabla^{1-T_A^L}\right)^{\overline{\omega}_2} + \\ &\left(1 - \nabla^{1-T_A^L}\right)^{\overline{\omega}_2} - \left(1 - \nabla^{1-T_A^L}\right)^{\overline{\omega}_1} \left(1 - \nabla^{1-T_A^L}\right)^{\overline{\omega}_2}, \\ &1 - \left(1 - \nabla^{1-T_A^U}\right)^{\overline{\omega}_1} + \left(1 - \nabla^{1-T_A^U}\right)^{\overline{\omega}_1} - \left(1 - \nabla^{1-T_A^U}\right)^{\overline{\omega}_2} + \\ &\left(1 - \nabla^{1-T_A^U}\right)^{\overline{\omega}_2} - \left(1 - \nabla^{1-T_A^U}\right)^{\overline{\omega}_1} \left(1 - \nabla^{1-T_A^U}\right)^{\overline{\omega}_2} \end{aligned} \right], \right. \\
&\quad \left[\begin{aligned} &1 - \left(1 - \nabla^{1-I_A^L}\right)^{\overline{\omega}_1} + \left(1 - \nabla^{1-I_A^L}\right)^{\overline{\omega}_1} - \left(1 - \nabla^{1-I_A^L}\right)^{\overline{\omega}_2} + \\ &\left(1 - \nabla^{1-I_A^L}\right)^{\overline{\omega}_2} - \left(1 - \nabla^{1-I_A^L}\right)^{\overline{\omega}_1} \left(1 - \nabla^{1-I_A^L}\right)^{\overline{\omega}_2}, \\ &1 - \left(1 - \nabla^{1-I_A^U}\right)^{\overline{\omega}_1} + \left(1 - \nabla^{1-I_A^U}\right)^{\overline{\omega}_1} - \left(1 - \nabla^{1-I_A^U}\right)^{\overline{\omega}_2} + \\ &\left(1 - \nabla^{1-I_A^U}\right)^{\overline{\omega}_2} - \left(1 - \nabla^{1-I_A^U}\right)^{\overline{\omega}_1} \left(1 - \nabla^{1-I_A^U}\right)^{\overline{\omega}_2} \end{aligned} \right], \\
&\quad \left[\begin{aligned} &\left(1 - \nabla^{F_A^L}\right)^{\overline{\omega}_1} \left(1 - \nabla^{F_A^L}\right)^{\overline{\omega}_2}, \left(1 - \nabla^{F_A^U}\right)^{\overline{\omega}_1} \left(1 - \nabla^{F_A^U}\right)^{\overline{\omega}_2} \end{aligned} \right], \\
&\quad \left(1 - \nabla^{T_A}\right)^{\overline{\omega}_1} \left(1 - \nabla^{T_A}\right)^{\overline{\omega}_2}, \left(1 - \nabla^{I_A}\right)^{\overline{\omega}_1} \left(1 - \nabla^{I_A}\right)^{\overline{\omega}_2}, \\
&\quad 1 - \left(1 - \nabla^{1-F_A}\right)^{\overline{\omega}_1} + \left(1 - \nabla^{1-F_A}\right)^{\overline{\omega}_1} - \\
&\quad \left(1 - \nabla^{1-F_A}\right)^{\overline{\omega}_2} \left(1 - \nabla^{1-F_A}\right)^{\overline{\omega}_2} - \left(1 - \nabla^{1-F_A}\right)^{\overline{\omega}_1} \left(1 - \nabla^{1-F_A}\right)^{\overline{\omega}_2} \left. \right\}, \\
&= \left\{ \left(\left[\begin{aligned} &1 - \left(1 - \nabla^{1-T_A^L}\right)^{\overline{\omega}_1}, 1 - \left(1 - \nabla^{1-T_A^U}\right)^{\overline{\omega}_1} \end{aligned} \right], \right. \right. \\
&\quad \left[\begin{aligned} &1 - \left(1 - \nabla^{1-I_A^L}\right)^{\overline{\omega}_1}, 1 - \left(1 - \nabla^{1-I_A^U}\right)^{\overline{\omega}_1} \end{aligned} \right], \\
&\quad \left[\begin{aligned} &\left(1 - \nabla^{F_A^L}\right)^{\overline{\omega}_1}, \left(1 - \nabla^{F_A^U}\right)^{\overline{\omega}_1} \end{aligned} \right], \\
&\quad \left(1 - \nabla^{T_A}\right)^{\overline{\omega}_1}, \left(1 - \nabla^{I_A}\right)^{\overline{\omega}_1}, \\
&\quad \left. 1 - \left(1 - \nabla^{1-F_A}\right)^{\overline{\omega}_1} \right) \oplus \left(\left[\begin{aligned} &1 - \left(1 - \nabla^{1-T_A^L}\right)^{\overline{\omega}_2}, 1 - \left(1 - \nabla^{1-T_A^U}\right)^{\overline{\omega}_2} \end{aligned} \right], \right. \\
&\quad \left[\begin{aligned} &1 - \left(1 - \nabla^{1-I_A^L}\right)^{\overline{\omega}_2}, 1 - \left(1 - \nabla^{1-I_A^U}\right)^{\overline{\omega}_2} \end{aligned} \right], \\
&\quad \left[\begin{aligned} &\left(1 - \nabla^{F_A^L}\right)^{\overline{\omega}_2}, \left(1 - \nabla^{F_A^U}\right)^{\overline{\omega}_2} \end{aligned} \right], \\
&\quad \left(1 - \nabla^{T_A}\right)^{\overline{\omega}_2}, \left(1 - \nabla^{I_A}\right)^{\overline{\omega}_2}, \\
&\quad \left. 1 - \left(1 - \nabla^{1-F_A}\right)^{\overline{\omega}_2} \right) \left. \right\} \\
&= \overline{\omega}_1 \nabla^A \oplus \overline{\omega}_2 \nabla^A
\end{aligned}$$

$$\begin{aligned}
\triangleright \quad \left((\nabla^A)^{\varpi_1} \right)^{\varpi_2} &= \left(\left[\left(\nabla^{1-T_A^L} \right)^{\varpi_1}, \left(\nabla^{1-T_A^U} \right)^{\varpi_1} \right], \left[\left(\nabla^{1-I_A^L} \right)^{\varpi_1}, \left(\nabla^{1-I_A^U} \right)^{\varpi_1} \right], \right. \\
&\quad \left[1 - \left(1 - \nabla^{F_A^L} \right)^{\varpi_1}, 1 - \left(1 - \nabla^{F_A^U} \right)^{\varpi_1} \right], \\
&\quad \left. 1 - \left(1 - \nabla^{T_A} \right)^{\varpi_1}, 1 - \left(1 - \nabla^{I_A} \right)^{\varpi_1}, \left(\nabla^{1-F_A} \right)^{\varpi_1} \right)^{\varpi_2} \\
&= \left(\left[\left(\nabla^{1-T_A^L} \right)^{\varpi_1 \varpi_2}, \left(\nabla^{1-T_A^U} \right)^{\varpi_1 \varpi_2} \right], \left[\left(\nabla^{1-I_A^L} \right)^{\varpi_1 \varpi_2}, \left(\nabla^{1-I_A^U} \right)^{\varpi_1 \varpi_2} \right], \right. \\
&\quad \left[1 - \left(1 - \left(1 - \left(1 - \nabla^{F_A^L} \right)^{\varpi_1} \right)^{\varpi_2}, 1 - \left(1 - \left(1 - \left(1 - \nabla^{F_A^U} \right)^{\varpi_1} \right)^{\varpi_2} \right)^{\varpi_2} \right], \right. \\
&\quad \left. 1 - \left(1 - \left(1 - \left(1 - \nabla^{T_A} \right)^{\varpi_1} \right)^{\varpi_2} \right)^{\varpi_2}, \right. \\
&\quad \left. 1 - \left(1 - \left(1 - \left(1 - \nabla^{I_A} \right)^{\varpi_1} \right)^{\varpi_2} \right)^{\varpi_2}, \left(\left(\nabla^{1-F_A} \right)^{\varpi_1} \right)^{\varpi_2} \right)^{\varpi_2} \\
&= \left(\left[\left(\nabla^{1-T_A^L} \right)^{\varpi_1 \varpi_2}, \left(\nabla^{1-T_A^U} \right)^{\varpi_1 \varpi_2} \right], \left[\left(\nabla^{1-I_A^L} \right)^{\varpi_1 \varpi_2}, \left(\nabla^{1-I_A^U} \right)^{\varpi_1 \varpi_2} \right], \right. \\
&\quad \left[1 - \left(1 - \nabla^{F_A^L} \right)^{\varpi_1 \varpi_2}, 1 - \left(1 - \nabla^{F_A^U} \right)^{\varpi_1 \varpi_2} \right], \\
&\quad \left. 1 - \left(1 - \nabla^{T_A} \right)^{\varpi_1 \varpi_2}, 1 - \left(1 - \nabla^{I_A} \right)^{\varpi_1 \varpi_2}, \left(\nabla^{1-F_A} \right)^{\varpi_1 \varpi_2} \right)^{\varpi_2} \\
&= \left(\nabla^A \right)^{\varpi_1 \varpi_2}
\end{aligned}$$

Definition 3.9 Let $A = \left(\left[T_A^L, T_A^U \right], \left[I_A^L, I_A^U \right], \left[F_A^L, F_A^U \right], T_A, I_A, F_A \right)$, $B = \left(\left[T_B^L, T_B^U \right], \left[I_B^L, I_B^U \right], \left[F_B^L, F_B^U \right], T_B, I_B, F_B \right)$ be two NCN, then $\tilde{d} = [A, B]$ is referred as DNCN.

Definition 3.10 Let $A = \left(\left[T_A^L, T_A^U \right], \left[I_A^L, I_A^U \right], \left[F_A^L, F_A^U \right], T_A, I_A, F_A \right)$ be a NCN the exponential law for interval value parameter for $\tilde{\nabla} = \left[\nabla^L, \nabla^U \right]$ is defined as

$$(\tilde{\nabla})^A = \left\{ \left\{ \left(\begin{array}{l} [(\nabla^L)^{1-(T_A)^L}, (\nabla^L)^{1-(T_A)^U}], \\ [(\nabla^L)^{1-(I_A)^L}, (\nabla^L)^{1-(I_A)^U}], \\ [1 - (\nabla^L)^{(F_A)^L}, 1 - (\nabla^L)^{(F_A)^U}], \\ 1 - (\nabla^L)^{T_A}, 1 - (\nabla^L)^{I_A}, (\nabla^L)^{1-F_A} \end{array} \right), \right. \right. \\ \left. \left\{ \begin{array}{l} [(\nabla^U)^{1-(T_A)^L}, (\nabla^U)^{1-(T_A)^U}], \\ [(\nabla^U)^{1-(I_A)^L}, (\nabla^U)^{1-(I_A)^U}], \\ [1 - (\nabla^U)^{(F_A)^L}, 1 - (\nabla^U)^{(F_A)^U}], \\ 1 - (\nabla^U)^{T_A}, 1 - (\nabla^U)^{I_A}, (\nabla^U)^{1-F_A} \end{array} \right) \right\}, \nabla^L, \nabla^U \in [0, 1] \right\} \\ \left\{ \left\{ \left(\begin{array}{l} [(1/(\nabla^L))^{1-(T_A)^L}, (1/(\nabla^L))^{1-(T_A)^U}], \\ [(1/(\nabla^L))^{1-(I_A)^L}, (1/(\nabla^L))^{1-(I_A)^U}], \\ [1 - (1/(\nabla^L))^{(F_A)^L}, 1 - (1/(\nabla^L))^{(F_A)^U}], \\ 1 - (1/(\nabla^L))^{T_A}, 1 - (1/(\nabla^L))^{I_A}, (1/(\nabla^L))^{1-F_A} \end{array} \right), \right. \right. \\ \left. \left\{ \begin{array}{l} [(1/(\nabla^U))^{1-(T_A)^L}, (1/(\nabla^U))^{1-(T_A)^U}], \\ [(1/(\nabla^U))^{1-(I_A)^L}, (1/(\nabla^U))^{1-(I_A)^U}], \\ [1 - (1/(\nabla^U))^{(F_A)^L}, 1 - (1/(\nabla^U))^{(F_A)^U}], \\ 1 - (1/(\nabla^U))^{T_A}, 1 - (1/(\nabla^U))^{I_A}, (1/(\nabla^U))^{1-F_A} \end{array} \right) \right\}, \nabla^L, \nabla^U > 1 \right\}$$

In both cases it is neutrosophic cubic dual.

Example 3.11 Let $A = ([0.3, 0.7], [0.2, 0.7], [0.3, 0.8], 0.5, 0.5, 0.6)$ be a NCN and $\tilde{\nabla} = [0.3, 0.7]$, $\tilde{\nabla} = [4, 8]$, then

$$(\tilde{\nabla})^A = \left\{ \left(\begin{array}{l} [0.430, 0.696], [0.381, 0.696], \\ [0.303, 0.618], 0.514, 0.452, 0.617 \end{array} \right), \tilde{\nabla} = [0.3, 0.7] \right\} \\ \left\{ \left(\begin{array}{l} [0.779, 0.898], [0.751, 0.989], \\ [0.101, 0.248], 0.807, 0.836, 0.867 \end{array} \right), \tilde{\nabla} = [4, 8] \right\} \\ \left\{ \left(\begin{array}{l} [0.378, 0.659], [0.329, 0.659], \\ [0.340, 0.670], 0.564, 0.500, 0.574 \end{array} \right), \tilde{\nabla} = [4, 8] \right\} \\ \left\{ \left(\begin{array}{l} [0.233, 0.535], [0.189, 0.535], \\ [0.464, 0.810], 0.712, 0.646, 0.435 \end{array} \right), \tilde{\nabla} = [4, 8] \right\}$$

Definition 3.12 Let $\tilde{d}_i = [A_i, B_i], (i = 1, 2)$ be two DNCN and ϖ be a real number then the algebraic operations are defined as

- $\tilde{d}_i \oplus \tilde{d}_j = [A_i \oplus A_j, B_i \oplus B_j]$
- $\tilde{d}_i \otimes \tilde{d}_j = [A_i \otimes A_j, B_i \otimes B_j]$
- $\varpi \tilde{d}_1 = [\varpi A_1, \varpi B_1]$
- $(\tilde{d}_1)^\varpi = [(A_1)^\varpi, (B_1)^\varpi]$

In these operations, first we use an interval exponential operational laws and then operation between NCNs or

real numbers are used. Hence it provide both the rationality of interval exponential operational laws and NCNs or real number operational laws as well.

Definition 3.13 Let $\tilde{d} = \left\{ \left(\left[T_{A^L}^L, T_{A^L}^U \right], \left[I_{A^L}^L, I_{A^L}^U \right], \left[F_{A^L}^L, F_{A^L}^U \right], T_{A^L}, I_{A^L}, F_{A^L} \right), \left(\left[T_{A^U}^L, T_{A^U}^U \right], \left[I_{A^U}^L, I_{A^U}^U \right], \left[F_{A^U}^L, F_{A^U}^U \right], T_{A^U}, I_{A^U}, F_{A^U} \right) \right\}$ be a

DNCN and $\tilde{d}^* = \left\{ ([1,1], [1,1], [0,0], 0,0,1), ([1,1], [1,1], [0,0], 0,0,1) \right\}$ be the maximum DNCN, then the

cosine measure(C_m) is defined as

$$C_m(\tilde{d}) = \left\{ \cos \frac{\pi}{36} \left(1 - T_{A^L}^L + 1 - T_{A^L}^U + 1 - I_{A^L}^L + 1 - I_{A^L}^U + F_{A^L}^L + F_{A^L}^U + T_{A^L} + I_{A^L} + 1 - F_{A^L} + 1 - T_{A^U}^L + 1 - T_{A^U}^U + 1 - I_{A^U}^L + 1 - I_{A^U}^U + F_{A^U}^L + F_{A^U}^U + T_{A^U} + I_{A^U} + 1 - F_{A^U} \right) \right\},$$

$$C_m(\tilde{d}) \in [0,1]$$

Definition 3.14 Let $C_m(\tilde{d}_1)$ and $C_m(\tilde{d}_2)$ be the cosine measures of two NCN then

$$C_m(\tilde{d}_1) > C_m(\tilde{d}_2) \Rightarrow \tilde{d}_1 > \tilde{d}_2 \text{ and } C_m(\tilde{d}_1) = C_m(\tilde{d}_2) \Rightarrow \tilde{d}_1 = \tilde{d}_2.$$

4. Neutrosophic Cubic Exponential Weighted Aggregation operator

Using definitions 3.1 and 3.10, in this section we propose the NCWEA and DNCWEA operators, where the base is crisp value or an interval numbers and the exponent is a NCNs.

Definition 4.1 We define the Neutrosophic cubic weighted exponential averaging operator (NCWEA) as

$$NCWEA(N_1, N_2, \dots, N_m) = \bigotimes_{i=1}^m (\nabla_i)^{N_i}$$

where N_i ($i = 1, 2, \dots, m$) are weight and ∇_i ($i = 1, 2, \dots, m$) real numbers respectively.

Theorem 4.2 Let $N_i = \left(\left[T_{N_i}^L, T_{N_i}^U \right], \left[I_{N_i}^L, I_{N_i}^U \right], \left[F_{N_i}^L, F_{N_i}^U \right], T_{N_i}, I_{N_i}, F_{N_i} \right)$ for ($i = 1, 2, \dots, m$) be the collection

of NCs and ∇_i ($i = 1, 2, \dots, m$) are real numbers respectively, then the NCWEA is a NCs, where

$$NCWEA(N_1, N_2, \dots, N_m) = \begin{cases} \begin{pmatrix} \left[\bigotimes_{i=1}^m (\nabla_i)^{1-T_{N_i}^L}, \bigotimes_{i=1}^m (\nabla_i)^{1-T_{N_i}^U} \right], \\ \left[\bigotimes_{i=1}^m (\nabla_i)^{1-I_{N_i}^L}, \bigotimes_{i=1}^m (\nabla_i)^{1-I_{N_i}^U} \right], \\ \left[1 - \bigotimes_{i=1}^m (\nabla_i)^{F_{N_i}^L}, 1 - \bigotimes_{i=1}^m (\nabla_i)^{F_{N_i}^U} \right], \\ \left(1 - \bigotimes_{i=1}^m (\nabla_i)^{T_{N_i}}, 1 - \bigotimes_{i=1}^m (\nabla_i)^{I_{N_i}}, \bigotimes_{i=1}^m (\nabla_i)^{1-F_{N_i}} \right) \end{pmatrix}, & \text{if } \nabla \in [0, 1] \\ \begin{pmatrix} \left[\bigotimes_{i=1}^m (1/\nabla_i)^{1-T_{N_i}^L}, \bigotimes_{i=1}^m (1/\nabla_i)^{1-T_{N_i}^U} \right], \\ \left[\bigotimes_{i=1}^m (1/\nabla_i)^{1-I_{N_i}^L}, \bigotimes_{i=1}^m (1/\nabla_i)^{1-I_{N_i}^U} \right], \\ \left[1 - \bigotimes_{i=1}^m (1/\nabla_i)^{F_{N_i}^L}, 1 - \bigotimes_{i=1}^m (1/\nabla_i)^{F_{N_i}^U} \right], \\ \left(1 - \bigotimes_{i=1}^m (1/\nabla_i)^{T_{N_i}}, 1 - \bigotimes_{i=1}^m (1/\nabla_i)^{I_{N_i}}, \bigotimes_{i=1}^m (1/\nabla_i)^{1-F_{N_i}} \right) \end{pmatrix}, & \text{if } \nabla > 1 \end{cases}$$

where N_i ($i = 1, 2, \dots, m$) is the weight of ∇_i ($i = 1, 2, \dots, m$).

Proof To prove the theorem we use mathematical induction, let $\nabla_i \in [0, 1]$ where $i = 1, 2, \dots, m$

For $m = 2$, we have

$$\begin{aligned} NCWEA(N_1, N_2) &= \bigotimes_{i=1}^2 (\nabla_i)^{N_i} \\ &= (\nabla_1)^{N_1} \otimes (\nabla_2)^{N_2} \\ &= \begin{pmatrix} \left[(\nabla_1)^{1-T_{N_1}^L}, (\nabla_1)^{1-T_{N_1}^U} \right], \\ \left[(\nabla_1)^{1-I_{N_1}^L}, (\nabla_1)^{1-I_{N_1}^U} \right], \\ \left[1 - (\nabla_1)^{F_{N_1}^L}, 1 - (\nabla_1)^{F_{N_1}^U} \right], \\ \left(1 - (\nabla_1)^{T_{N_1}}, 1 - (\nabla_1)^{I_{N_1}}, (\nabla_1)^{1-F_{N_1}} \right) \end{pmatrix} \otimes \begin{pmatrix} \left[(\nabla_2)^{1-T_{N_2}^L}, (\nabla_2)^{1-T_{N_2}^U} \right], \\ \left[(\nabla_2)^{1-I_{N_2}^L}, (\nabla_2)^{1-I_{N_2}^U} \right], \\ \left[1 - (\nabla_2)^{F_{N_2}^L}, 1 - (\nabla_2)^{F_{N_2}^U} \right], \\ \left(1 - (\nabla_2)^{T_{N_2}}, 1 - (\nabla_2)^{I_{N_2}}, (\nabla_2)^{1-F_{N_2}} \right) \end{pmatrix} \end{pmatrix}$$

$$\begin{aligned}
&= \left(\left[(\nabla_1)^{1-T_{N_1}^L} (\nabla_2)^{1-T_{N_2}^L}, (\nabla_1)^{1-T_{N_1}^U} (\nabla_2)^{1-T_{N_2}^U} \right], \left[(\nabla_1)^{1-I_{N_1}^L} (\nabla_2)^{1-I_{N_2}^L}, (\nabla_1)^{1-I_{N_1}^U} (\nabla_2)^{1-I_{N_2}^U} \right], \right. \\
&\quad \left[1 - (\nabla_1)^{F_{N_1}^L} + 1 - (\nabla_2)^{F_{N_2}^L} - \left(1 - (\nabla_1)^{F_{N_1}^L} \right) \left(1 - (\nabla_2)^{F_{N_2}^L} \right), \right. \\
&\quad \left. 1 - (\nabla_1)^{F_{N_1}^U} + 1 - (\nabla_2)^{F_{N_2}^U} - \left(1 - (\nabla_1)^{F_{N_1}^U} \right) \left(1 - (\nabla_2)^{F_{N_2}^U} \right) \right], \\
&\quad 1 - (\nabla_1)^{T_{N_1}} + 1 - (\nabla_2)^{T_{N_2}} - \left(1 - (\nabla_1)^{T_{N_1}} \right) \left(1 - (\nabla_2)^{T_{N_2}} \right), \\
&\quad \left. 1 - (\nabla_1)^{I_{N_1}} + 1 - (\nabla_2)^{I_{N_2}} - \left(1 - (\nabla_1)^{I_{N_1}} \right) \left(1 - (\nabla_2)^{I_{N_2}} \right), (\nabla_1)^{1-F_{N_1}} (\nabla_2)^{1-F_{N_2}} \right) \\
&= \left(\left[\bigotimes_{i=1}^2 (\nabla_i)^{1-T_{N_i}^L}, \bigotimes_{i=1}^2 (\nabla_i)^{1-T_{N_i}^L} \right], \left[\bigotimes_{i=1}^2 (\nabla_i)^{1-I_{N_i}^L}, \bigotimes_{i=1}^2 (\nabla_i)^{1-I_{N_i}^L} \right], \right. \\
&\quad \left[1 - (\nabla_1)^{F_{N_1}^L} + \left(1 - (\nabla_2)^{F_{N_2}^L} \right) (\nabla_1)^{F_{N_1}^L}, \right. \\
&\quad \left. 1 - (\nabla_1)^{F_{N_1}^U} + \left(1 - (\nabla_2)^{F_{N_2}^U} \right) (\nabla_1)^{F_{N_1}^U} \right], \\
&\quad 1 - (\nabla_1)^{T_{N_1}} + \left(1 - (\nabla_2)^{T_{N_2}} \right) (\nabla_1)^{T_{N_1}}, \\
&\quad \left. 1 - (\nabla_1)^{I_{N_1}} + \left(1 - (\nabla_2)^{I_{N_2}} \right) (\nabla_1)^{I_{N_1}}, \bigotimes_{i=1}^2 (\nabla_i)^{1-F_{N_i}} \right) \\
&= \left(\left[\bigotimes_{i=1}^2 (\nabla_i)^{1-T_{N_i}^L}, \bigotimes_{i=1}^2 (\nabla_i)^{1-T_{N_i}^L} \right], \left[\bigotimes_{i=1}^2 (\nabla_i)^{1-I_{N_i}^L}, \bigotimes_{i=1}^2 (\nabla_i)^{1-I_{N_i}^L} \right], \right. \\
&\quad \left[1 - (\nabla_1)^{F_{N_1}^L} (\nabla_2)^{F_{N_2}^L}, 1 - (\nabla_1)^{F_{N_1}^U} (\nabla_2)^{F_{N_2}^U} \right], \\
&\quad \left. 1 - (\nabla_1)^{T_{N_1}} (\nabla_2)^{T_{N_2}}, 1 - (\nabla_1)^{I_{N_1}} (\nabla_2)^{I_{N_2}}, \bigotimes_{i=1}^2 (\nabla_i)^{1-F_{N_i}} \right) \\
&= \left(\left[\bigotimes_{i=1}^2 (\nabla_i)^{1-T_{N_i}^L}, \bigotimes_{i=1}^2 (\nabla_i)^{1-T_{N_i}^L} \right], \left[\bigotimes_{i=1}^2 (\nabla_i)^{1-I_{N_i}^L}, \bigotimes_{i=1}^2 (\nabla_i)^{1-I_{N_i}^L} \right], \right. \\
&\quad \left. \left[1 - \bigotimes_{i=1}^2 (\nabla_i)^{F_{N_i}^L}, 1 - \bigotimes_{i=1}^2 (\nabla_i)^{F_{N_i}^U} \right], 1 - \bigotimes_{i=1}^2 (\nabla_i)^{T_{N_i}}, 1 - \bigotimes_{i=1}^2 (\nabla_i)^{I_{N_i}}, \bigotimes_{i=1}^2 (\nabla_i)^{1-F_{N_i}} \right)
\end{aligned}$$

Assuming for $n = m$, is a NCs that is

$$NCWEA(N_1, N_2, \dots, N_m) = \left(\begin{array}{c} \left[\bigotimes_{i=1}^m (\nabla_i)^{1-T_{N_i}^L}, \bigotimes_{i=1}^m (\nabla_i)^{1-T_{N_i}^L} \right], \left[\bigotimes_{i=1}^m (\nabla_i)^{1-I_{N_i}^L}, \bigotimes_{i=1}^m (\nabla_i)^{1-I_{N_i}^L} \right], \\ \left[1 - \bigotimes_{i=1}^m (\nabla_i)^{F_{N_i}^L}, 1 - \bigotimes_{i=1}^m (\nabla_i)^{F_{N_i}^U} \right], \\ 1 - \bigotimes_{i=1}^m (\nabla_i)^{T_{N_i}}, 1 - \bigotimes_{i=1}^m (\nabla_i)^{I_{N_i}}, \bigotimes_{i=1}^m (\nabla_i)^{1-F_{N_i}} \end{array} \right)$$

We prove the result for $n = m + 1$, is a NCs.

$$NCWEA(N_1, N_2, \dots, N_{m+1}) = \left\{ \left(\begin{array}{c} \left[\bigotimes_{i=1}^m (\nabla_i)^{1-T_{N_i}^L}, \right. \\ \left. \bigotimes_{i=1}^m (\nabla_i)^{1-T_{N_i}^L} \right], \\ \left[\bigotimes_{i=1}^m (\nabla_i)^{1-I_{N_i}^L}, \right. \\ \left. \bigotimes_{i=1}^m (\nabla_i)^{1-I_{N_i}^L} \right], \\ \left[1 - \bigotimes_{i=1}^m (\nabla_i)^{F_{N_i}^L}, \right. \\ \left. 1 - \bigotimes_{i=1}^m (\nabla_i)^{F_{N_i}^U} \right], \\ 1 - \bigotimes_{i=1}^m (\nabla_i)^{T_{N_i}}, \\ 1 - \bigotimes_{i=1}^m (\nabla_i)^{I_{N_i}}, \\ \left. \bigotimes_{i=1}^m (\nabla_i)^{1-F_{N_i}} \right] \right\} \otimes \left\{ \begin{array}{c} \left[(\nabla_{\sigma+1})^{1-T_{N_{\sigma+1}}^L}, \right. \\ \left. (\nabla_{\sigma+1})^{1-T_{N_{\sigma+1}}^L} \right], \\ \left[(\nabla_{\sigma+1})^{1-I_{N_{\sigma+1}}^L}, \right. \\ \left. (\nabla_{\sigma+1})^{1-I_{N_{\sigma+1}}^L} \right], \\ \left[1 - (\nabla_{\sigma+1})^{F_{N_{\sigma+1}}^L}, \right. \\ \left. 1 - (\nabla_{\sigma+1})^{F_{N_{\sigma+1}}^U} \right], \\ 1 - (\nabla_{\sigma+1})^{T_{N_{\sigma+1}}}, \\ 1 - (\nabla_{\sigma+1})^{I_{N_{\sigma+1}}}, \\ \left. (\nabla_{\sigma+1})^{1-F_{N_{\sigma+1}}} \right] \end{array} \right\}$$

$$= \left(\begin{array}{c} \left[\bigotimes_{i=1}^{\sigma+1} (\nabla_i)^{1-T_{N_i}^L}, \bigotimes_{i=1}^{\sigma+1} (\nabla_i)^{1-T_{N_i}^L} \right], \\ \left[\bigotimes_{i=1}^{\sigma+1} (\nabla_i)^{1-I_{N_i}^L}, \bigotimes_{i=1}^{\sigma+1} (\nabla_i)^{1-I_{N_i}^L} \right], \\ \left[1 - \bigotimes_{i=1}^{\sigma+1} (\nabla_i)^{F_{N_i}^L}, 1 - \bigotimes_{i=1}^{\sigma+1} (\nabla_i)^{F_{N_i}^U} \right], \\ 1 - \bigotimes_{i=1}^{\sigma+1} (\nabla_i)^{T_{N_i}}, 1 - \bigotimes_{i=1}^{\sigma+1} (\nabla_i)^{I_{N_i}}, \bigotimes_{i=1}^{\sigma+1} (\nabla_i)^{1-F_{N_i}} \end{array} \right)$$

If $\nabla_i > 1$, then $0 < 1/\nabla_i < 1$, and then using above procedure a similar proof can be obtained for the following aggregation operators

$$NCWEA(N_1, N_2, \dots, N_m) = \left(\begin{array}{c} \left[\bigotimes_{i=1}^m (1/\nabla_i)^{1-T_{N_i}^L}, \bigotimes_{i=1}^m (1/\nabla_i)^{1-T_{N_i}^U} \right], \\ \left[\bigotimes_{i=1}^m (1/\nabla_i)^{1-I_{N_i}^L}, \bigotimes_{i=1}^m (1/\nabla_i)^{1-I_{N_i}^U} \right], \\ \left[1 - \bigotimes_{i=1}^m (1/\nabla_i)^{F_{N_i}^L}, 1 - \bigotimes_{i=1}^m (1/\nabla_i)^{F_{N_i}^U} \right], \\ \left[1 - \bigotimes_{i=1}^m (1/\nabla_i)^{T_{N_i}}, 1 - \bigotimes_{i=1}^m (1/\nabla_i)^{I_{N_i}}, \bigotimes_{i=1}^m (1/\nabla_i)^{1-F_{N_i}} \right] \end{array} \right)$$

This complete the proof.

Definition 4.3 Let $N_i = \left([T_{N_i}^L, T_{N_i}^U], [I_{N_i}^L, I_{N_i}^U], [F_{N_i}^L, F_{N_i}^U], T_{N_i}, I_{N_i}, F_{N_i} \right)$ for $(i = 1, 2, \dots, m)$ be the collection of NCs and $\tilde{\nabla} = [\nabla_i^L, \nabla_i^U] (i = 1, 2, \dots, m)$ be the collection of interval numbers, then the DNCWEA operator is defined as

$$DNCWEA(N_1, N_2, \dots, N_m) = \bigotimes_{i=1}^m (\tilde{\nabla}_i)^{N_i}$$

where $N_i (i = 1, 2, \dots, m)$ is the weight corresponding to $\tilde{\nabla}_i = [\nabla_i^L, \nabla_i^U] (i = 1, 2, \dots, m)$.

Theorem 4.4 Let $N_i = \left([T_{N_i}^L, T_{N_i}^U], [I_{N_i}^L, I_{N_i}^U], [F_{N_i}^L, F_{N_i}^U], T_{N_i}, I_{N_i}, F_{N_i} \right)$ for $(i = 1, 2, \dots, m)$ be the collection of NCs and $\tilde{\nabla} = [\nabla_i^L, \nabla_i^U] (i = 1, 2, \dots, m)$ be collection of interval numbers, then the DNCWEA operator is given by

$$NCWEA(N_1, N_2, \dots, N_m) =$$

$$\left\{ \left\{ \left(\begin{aligned} & \left[\bigotimes_{i=1}^m (\nabla_i^L)^{1-T_{N_i}^L}, \bigotimes_{i=1}^m (\nabla_i^L)^{1-T_{N_i}^L} \right], \\ & \left[\bigotimes_{i=1}^m (\nabla_i^L)^{1-I_{N_i}^L}, \bigotimes_{i=1}^m (\nabla_i^L)^{1-I_{N_i}^L} \right], \\ & \left[1 - \bigotimes_{i=1}^m (\nabla_i^L)^{F_{N_i}^L}, 1 - \bigotimes_{i=1}^m (\nabla_i^L)^{F_{N_i}^L} \right], \\ & 1 - \bigotimes_{i=1}^m (\nabla_i^L)^{T_{N_i}^L}, 1 - \bigotimes_{i=1}^m (\nabla_i^L)^{I_{N_i}^L}, \\ & \bigotimes_{i=1}^m (\nabla_i^L)^{1-F_{N_i}^L} \end{aligned} \right\}, \text{ if } 0 \leq \nabla_i^L \leq \nabla_i^U \leq 1 \right. \\ \left. \left\{ \left(\begin{aligned} & \left[\bigotimes_{i=1}^m (\nabla_i^U)^{1-T_{N_i}^L}, \bigotimes_{i=1}^m (\nabla_i^U)^{1-T_{N_i}^L} \right], \\ & \left[\bigotimes_{i=1}^m (\nabla_i^U)^{1-I_{N_i}^L}, \bigotimes_{i=1}^m (\nabla_i^U)^{1-I_{N_i}^L} \right], \\ & \left[1 - \bigotimes_{i=1}^m (\nabla_i^U)^{F_{N_i}^L}, 1 - \bigotimes_{i=1}^m (\nabla_i^U)^{F_{N_i}^L} \right], \\ & 1 - \bigotimes_{i=1}^m (\nabla_i^U)^{T_{N_i}^L}, 1 - \bigotimes_{i=1}^m (\nabla_i^U)^{I_{N_i}^L}, \\ & \bigotimes_{i=1}^m (\nabla_i^U)^{1-F_{N_i}^L} \end{aligned} \right\} \right. \\ \left. \left\{ \left(\begin{aligned} & \left[\bigotimes_{i=1}^m (1/\nabla_i^L)^{1-T_{N_i}^L}, \bigotimes_{i=1}^m (1/\nabla_i^L)^{1-T_{N_i}^L} \right], \\ & \left[\bigotimes_{i=1}^m (1/\nabla_i^L)^{1-I_{N_i}^L}, \bigotimes_{i=1}^m (1/\nabla_i^L)^{1-I_{N_i}^L} \right], \\ & \left[1 - \bigotimes_{i=1}^m (1/\nabla_i^L)^{F_{N_i}^L}, 1 - \bigotimes_{i=1}^m (1/\nabla_i^L)^{F_{N_i}^L} \right], \\ & 1 - \bigotimes_{i=1}^m (1/\nabla_i^L)^{T_{N_i}^L}, 1 - \bigotimes_{i=1}^m (1/\nabla_i^L)^{I_{N_i}^L}, \\ & \bigotimes_{i=1}^m (1/\nabla_i^L)^{1-F_{N_i}^L} \end{aligned} \right\}, \text{ if } \nabla_i^U \geq \nabla_i^L > 1 \right. \\ \left. \left\{ \left(\begin{aligned} & \left[\bigotimes_{i=1}^m (1/\nabla_i^U)^{1-T_{N_i}^L}, \bigotimes_{i=1}^m (1/\nabla_i^U)^{1-T_{N_i}^L} \right], \\ & \left[\bigotimes_{i=1}^m (1/\nabla_i^U)^{1-I_{N_i}^L}, \bigotimes_{i=1}^m (1/\nabla_i^U)^{1-I_{N_i}^L} \right], \\ & \left[1 - \bigotimes_{i=1}^m (1/\nabla_i^U)^{F_{N_i}^L}, 1 - \bigotimes_{i=1}^m (1/\nabla_i^U)^{F_{N_i}^L} \right], \\ & 1 - \bigotimes_{i=1}^m (1/\nabla_i^U)^{T_{N_i}^L}, \\ & 1 - \bigotimes_{i=1}^m (1/\nabla_i^U)^{I_{N_i}^L}, \bigotimes_{i=1}^m (1/\nabla_i^U)^{1-F_{N_i}^L} \end{aligned} \right\} \right\} \right\}$$

The proof is analogous to Theorem 4.2.

5. Decision making method based on the NCWEA and DNCWEA operators

Based on NCWEA and DNCWEA operators, a decision making problem can be dealt. In such MADM problem the weight is NCs and alternative value are crisp or interval numbers.

For this consider the MADM problem with m alternative $A = \{a_1, a_2, \dots, a_m\}$ and $C = \{y_1, y_2, \dots, y_n\}$ be n attributes. An expert has evaluated these attributes in the form of NCs and the suitable alternative in the form of crisp

value $\nabla_{ij} \in [0,1] (1/\nabla_{ij}, \text{if } \nabla_{ij} > 1)$ or interval numbers.

$$\nabla_{ij} = [\nabla_{ij}^L, \nabla_{ij}^U] \subseteq [0,1] (1/\nabla_{ij}^L, \text{if } \nabla_{ij}^L > 1; 1/\nabla_{ij}^U, \text{if } \nabla_{ij}^U > 1) \text{ for } (i = 1, 2, \dots, m; j = 1, 2, \dots, n).$$

Step 1: A preference value decision matrix $D = [\nabla_{ij}]$ or $D = [\tilde{\nabla}_{ij}]$ is constructed for m alternatives and n attributes, where weight is expressed as $N_j = ([T_{N_j}^L, T_{N_j}^U], [I_{N_j}^L, I_{N_j}^U], [F_{N_j}^L, F_{N_j}^U], T_{N_j}, I_{N_j}, F_{N_j})$ ($j = 1, 2, \dots, n$) in NCs form for corresponding attributes.

Step 2: Using the suitable aggregation operator like NCEWA or DNCEWA the overall aggregated value is obtained.

Step 3: Using the measurement function of definition 3.3 or definition 3.14, values are ranked.

Step 4: The best alternative is chosen amongst the ranked.

6. Example

Next we provide an illustrative example as an application to our aggregation operators.

Example 6.1

Our example from daily life is an application to pick the best alternative using the decision making matrix with base either crisp values or interval numbers and weight as neutrosophic cubic number.

A steering committee is interested to prioritize the set of information improvement project using a multi-attribute decision making method. The committee must prioritized the implementation and development of set of six information technologies improvement projects a_j ($j = 1, 2, \dots, 6$). The weight of these six attributes are expressed

in term of NCs

$$\{a_1, a_2, a_3, a_4, a_5, a_6\} = \left\{ ([0.5, 0.6], [0.2, 0.5], [0.4, 0.8], 0.7, 0.8, 0.4), ([0.2, 0.5], [0.7, 0.9], [0.3, 0.7], 0.8, 0.5, 0.3), \right. \\ \left. ([0.4, 0.7], [0.2, 0.5], [0.5, 0.7], 0.3, 0.6, 0.2), ([0.3, 0.6], [0.4, 0.7], [0.2, 0.5], 0.6, 0.4, 0.7), \right. \\ \left. ([0.2, 0.5], [0.3, 0.7], [0.2, 0.6], 0.5, 0.3, 0.8), ([0.1, 0.6], [0.3, 0.6], [0.4, 0.8], 0.6, 0.9, 0.4) \right\}$$

by decision maker. The three factors (alternatives) $\{y_1, y_2, y_3\}$, y_1 - productivity to maximize the efficiency and effectiveness, y_2 differentiation from products and services of competitors, and y_3 management to assist the managers in enhancing the planning, are considered to assess the contribution of these project. The goal of committee is to choose best alternative among them.

Step 1: The decision maker(s) is (are) required to make the suitable judgement of alternatives y_i ($i = 1, 2, 3$) with respect to these attributes a_j ($j = 1, 2, \dots, 6$) and give the evaluated information of the crisp values $\nabla_{ij} \in [0,1]$, which is structured as follow.

$$D = [\nabla_{ij}] = \begin{bmatrix} 0.7 & 0.6 & 0.5 & 0.8 & 0.5 & 0.4 \\ 0.5 & 0.7 & 0.9 & 0.4 & 0.6 & 0.7 \\ 0.2 & 0.3 & 0.6 & 0.5 & 0.7 & 0.6 \end{bmatrix}$$

Step 2: Utilizing the NCEWA operator to evaluate these preferences of alternatives

$$\begin{aligned}
 d_1 = NCWEA(a_1, a_2, \dots, a_6) &= \left(\begin{array}{c} \left[\bigotimes_{j=1}^6 (\nabla_{1j})^{1-T_{N_j}^L}, \bigotimes_{j=1}^6 (\nabla_{1j})^{1-T_{N_j}^U} \right], \\ \left[\bigotimes_{j=1}^6 (\nabla_{1j})^{1-I_{N_j}^L}, \bigotimes_{j=1}^6 (\nabla_{1j})^{1-I_{N_j}^U} \right], \\ \left[1 - \bigotimes_{j=1}^6 (\nabla_{1j})^{F_{N_j}^L}, 1 - \bigotimes_{j=1}^6 (\nabla_{1j})^{F_{N_j}^U} \right], \\ \left[1 - \bigotimes_{j=1}^6 (\nabla_{1j})^{T_{N_j}}, 1 - \bigotimes_{j=1}^6 (\nabla_{1j})^{I_{N_j}}, \bigotimes_{j=1}^6 (\nabla_{1j})^{1-F_{N_j}} \right] \end{array} \right) \\
 &= ([0.0790, 0.2445], [0.0874, 0.2959], [0.6964, 0.9082], 0.8499, 0.8748, 0.1523) \\
 \text{Similarly } d_2 &= ([0.1266, 0.2859], [0.1491, 0.3657], [0.5789, 0.8545], 0.8381, 0.8054, 0.2616) \\
 d_3 &= ([0.0367, 0.1275], [0.0459, 0.1827], [0.8126, 0.9684], 0.9568, 0.9521, 0.0606)
 \end{aligned}$$

Step 3: We have $C_m(d_1) = 0.9998067$, $C_m(d_2) = 0.9998217$ and $C_m(d_3) = 0.9997450$, here $C_m(d_2) > C_m(d_1) > C_m(d_3)$ the ranking order of alternatives are $y_2 > y_1 > y_3$.

Step 4: The best alternative on the basis of these calculations is y_2 management to assist the managers in improving their planning.

If the suitable judgment of each attribute y_i ($i = 1, 2, 3$) is made for interval numbers of interval valued decision making matrix:

In such a case the proposed MADM is based on DNCWEA operator may be applied to choose the suitable alternative, described in the following steps:

Step 1: First of all the interval valued decision making matrix is formed by decision maker:

$$\tilde{D} = (\tilde{V}_{ij}) = \begin{bmatrix} [0.5, 0.8] & [0.4, 0.6] & [0.2, 0.5] & [0.7, 0.9] & [0.4, 0.6] & [0.3, 0.4] \\ [0.4, 0.5] & [0.6, 0.8] & [0.8, 0.9] & [0.4, 0.6] & [0.5, 0.6] & [0.7, 0.8] \\ [0.2, 0.3] & [0.3, 0.5] & [0.5, 0.7] & [0.4, 0.5] & [0.6, 0.8] & [0.5, 0.8] \end{bmatrix}$$

Step 2: The proposed MADM based on DNCWEA operator is applied to decision making matrix considering NC values of attributes a_j ($j = 1, 2, \dots, 6$) as weight for alternatives y_i ($i = 1, 2, 3$).

$$\begin{aligned}
\tilde{d}_1 = NCWEA(a_1, a_2, \dots, a_6) &= \left(\left(\begin{aligned} &\left[\bigotimes_{j=1}^6 (\nabla_{1j}^L)^{1-T_{N_j}^L}, \bigotimes_{j=1}^6 (\nabla_{1j}^L)^{1-T_{N_j}^U} \right], \\ &\left[\bigotimes_{j=1}^6 (\nabla_{1j}^L)^{1-I_{N_j}^L}, \bigotimes_{j=1}^6 (\nabla_{1j}^L)^{1-I_{N_j}^U} \right], \\ &\left[1 - \bigotimes_{j=1}^6 (\nabla_{1j}^L)^{F_{N_j}^L}, 1 - \bigotimes_{j=1}^6 (\nabla_{1j}^L)^{F_{N_j}^U} \right], \\ &1 - \bigotimes_{j=1}^6 (\nabla_{1j}^L)^{T_{N_j}}, 1 - \bigotimes_{j=1}^6 (\nabla_{1j}^L)^{I_{N_j}}, \bigotimes_{j=1}^6 (\nabla_{1j}^L)^{1-F_{N_j}} \end{aligned} \right) \right) \\
&\left(\left(\begin{aligned} &\left[\bigotimes_{j=1}^6 (\nabla_{1j}^U)^{1-T_{N_j}^L}, \bigotimes_{j=1}^6 (\nabla_{1j}^U)^{1-T_{N_j}^U} \right], \\ &\left[\bigotimes_{j=1}^6 (\nabla_{1j}^U)^{1-I_{N_j}^L}, \bigotimes_{j=1}^6 (\nabla_{1j}^U)^{1-I_{N_j}^U} \right], \\ &\left[1 - \bigotimes_{j=1}^6 (\nabla_{1j}^U)^{F_{N_j}^L}, 1 - \bigotimes_{j=1}^6 (\nabla_{1j}^U)^{F_{N_j}^U} \right], \\ &1 - \bigotimes_{j=1}^6 (\nabla_{1j}^U)^{T_{N_j}}, 1 - \bigotimes_{j=1}^6 (\nabla_{1j}^U)^{I_{N_j}}, \bigotimes_{j=1}^6 (\nabla_{1j}^U)^{1-F_{N_j}} \end{aligned} \right) \right) \\
&= \left\{ ([0.0163, 0.1001], [0.0220, 0.1216], [0.8766, 0.9819], 0.9547, 0.9691, 0.0348), \right. \\
&\quad \left. ([0.1061, 0.2962], [0.1424, 0.3700], [0.1524, 0.4290], 0.2675, 0.8458, 0.1773) \right\} \\
\tilde{d}_2 &= \left\{ ([0.0835, 0.2133], [0.09542, 0.2875], [0.6657, 0.9098], 0.8921, 0.8609, 0.1802) \right\} \\
&\quad \left\{ ([0.2221, 0.3792], [0.2173, 0.4416], [0.4986, 0.7823], 0.7512, 0.7240, 0.2018) \right\} \\
\tilde{d}_3 &= \left\{ ([0.0211, 0.0951], [0.0274, 0.1384], [0.8525, 0.9804], 0.9703, 0.9682, 0.0426) \right\} \\
&\quad \left\{ ([0.1069, 0.2433], [0.1125, 0.2970], [0.7165, 0.9052], 0.8853, 0.8736, 0.1526) \right\}
\end{aligned}$$

Step 3: To rank the value the cosine measure is determined for the values computed in Step 2,

$$C_m(\tilde{d}_1) = 0.3346, C_m(\tilde{d}_2) = 0.3613, C_m(\tilde{d}_3) = 0.2460. \text{ The ranked alternatives are as } y_2 > y_1 > y_3.$$

Step 4: The best alternative on the basis of these calculations is y_2 management to assist the managers in improving their planning.

7. Validity Test

Wang and Triantaphyllou [33] proposed criteria to figure out the validity of a MADM method.

Test Criterion 1: "The replacement of a non-optimal alternative with an arbitrary worse value does not change the index of best alternative".

Test Criterion 2: "The transitive property is satisfied by an effective MADM method".

Test Criterion 3: "If MADM problem is decomposed into the sub DM problem and the same MADM procedure is applied to sub problem for ranking of alternatives, the order ranking of the alternatives must be similar to ranking of un decomposed DM problem".

Validity Test by Criterion 1

We change the rating value of non-optimal alternative y_3 by $y_3' = [0.3 \ 0.8 \ 0.4 \ 0.2 \ 0.9 \ 0.5]$, we have $d_3 = ([0.0307, 0.1155], [0.0272, 0.1118], [0.8034, 0.9585], 0.9348, 0.9462, 0.0579)$ and $C_m(d_3) = 0.1695$, made no changes in said method.

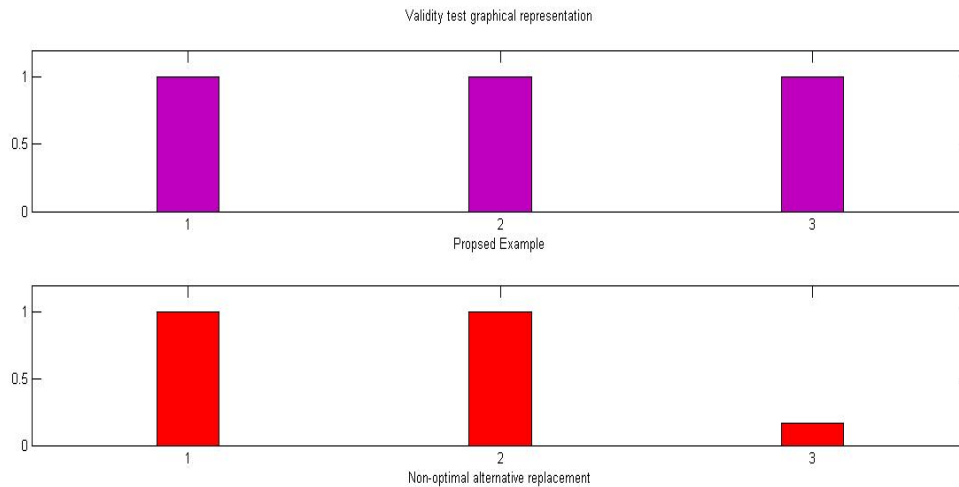


Fig.1: Comparison of graph by changing an alternative with non-optimal alternative.

The graph indicates that if the non-optimal value does not cause any change in optimal alternative, which is 1 (y_1) in this graph and 3 (y_3) is non-optimal alternative.

Validity Test by Criterion 2

Under this criterion we decompose the decision matrix into $\{y_1, y_2\}$, $\{y_1, y_3\}$ and $\{y_2, y_3\}$, we observe that $y_2 > y_1$, $y_1 > y_3$ and $y_2 > y_3$. That is transitive property is satisfied.

Validity Test by Criterion 3

By validity test in criterion 2, we observe that the sub DM satisfy the original ranking order, that is $y_2 > y_1 > y_3$.

Hence validity test 1,2 and 3 are satisfied by MADM.

8. Comparison Analysis

In this section we compare the exponential aggregation operator with NCWA operator.

$$d'_1 = NCWA(a_1, a_2, \dots, a_6) = \left(\begin{array}{c} \left[1 - \bigotimes_{j=1}^6 (1 - T_{N_j}^L)^{w_j}, 1 - \bigotimes_{j=1}^6 (1 - T_{N_j}^U)^{w_j} \right], \\ \left[1 - \bigotimes_{j=1}^6 (1 - I_{N_j}^L)^{w_j}, 1 - \bigotimes_{j=1}^6 (1 - I_{N_j}^U)^{w_j} \right], \\ \left[\bigotimes_{j=1}^6 (F_{N_j}^L)^{w_j}, \bigotimes_{j=1}^6 (F_{N_j}^U)^{w_j} \right], \\ \left(\bigotimes_{j=1}^6 (T_{N_j})^{w_j}, \bigotimes_{j=1}^6 (I_{N_j})^{w_j}, 1 - \bigotimes_{j=1}^6 (1 - (F_{N_j})_F)^{w_j} \right) \end{array} \right)$$

$$d'_1 = ([0.7311, 0.9551], [0.8277, 0.9352], [0.0154, 0.2351], 0.1429, 0.1102, 0.9072)$$

$$d'_2 = ([0.7309, 0.9682], [0.8384, 0.9880], [0.0153, 0.2411], 0.0910, 0.1086, 0.9062)$$

$$d'_3 = ([0.5974, 0.9262], [0.7160, 0.9608], [0.0334, 0.3001], 0.1484, 0.1461, 0.9072)$$

In this case $C_m(d'_1) = 0.9831$, $C_m(d'_2) = 0.9851$, and $C_m(d'_3) = 0.9663$. This yields that

$$y_2 > y_1 > y_3.$$

Here the role of real number (base) is change by weight in neutrosophic cubic weighted aggregation operator (NCWA) operator, we observe the same result as by NCEWA operator.

Now we compare the exponential aggregation operator with NCWA operator.

$$d'_1 = NCWA(a_1, a_2, \dots, a_6) = \left(\begin{array}{c} \left[\frac{\bigotimes_{j=1}^6 (1 + T_{N_j}^L)^{w_j} - \bigotimes_{j=1}^6 (1 - T_{N_j}^L)^{w_j}}{\bigotimes_{j=1}^6 (1 + T_{N_j}^L)^{w_j} + \bigotimes_{j=1}^6 (1 - T_{N_j}^L)^{w_j}}, \frac{\bigotimes_{j=1}^6 (1 + T_{N_j}^U)^{w_j} - \bigotimes_{j=1}^6 (1 - T_{N_j}^U)^{w_j}}{\bigotimes_{j=1}^6 (1 + T_{N_j}^U)^{w_j} + \bigotimes_{j=1}^6 (1 - T_{N_j}^U)^{w_j}} \right], \left[\frac{\bigotimes_{j=1}^6 (1 + I_{N_j}^L)^{w_j} - \bigotimes_{j=1}^6 (1 - I_{N_j}^L)^{w_j}}{\bigotimes_{j=1}^6 (1 + I_{N_j}^L)^{w_j} + \bigotimes_{j=1}^6 (1 - I_{N_j}^L)^{w_j}}, \frac{\bigotimes_{j=1}^6 (1 + I_{N_j}^U)^{w_j} - \bigotimes_{j=1}^6 (1 - I_{N_j}^U)^{w_j}}{\bigotimes_{j=1}^6 (1 + I_{N_j}^U)^{w_j} + \bigotimes_{j=1}^6 (1 - I_{N_j}^U)^{w_j}} \right], \\ \left[\frac{2 \bigotimes_{j=1}^6 (F_{N_j}^L)^{w_j}}{\bigotimes_{j=1}^6 (2 - F_{N_j}^L)^{w_j} + \bigotimes_{j=1}^6 (F_{N_j}^L)^{w_j}}, \frac{2 \bigotimes_{j=1}^6 (F_{N_j}^U)^{w_j}}{\bigotimes_{j=1}^6 (2 - F_{N_j}^U)^{w_j} + \bigotimes_{j=1}^6 (F_{N_j}^U)^{w_j}} \right], \\ \left(\frac{2 \bigotimes_{j=1}^6 (T_{N_j})^{w_j}}{\bigotimes_{j=1}^6 (2 - T_{N_j})^{w_j} + \bigotimes_{j=1}^6 (T_{N_j})^{w_j}}, \frac{2 \bigotimes_{j=1}^6 (I_{N_j})^{w_j}}{\bigotimes_{j=1}^6 (2 - I_{N_j})^{w_j} + \bigotimes_{j=1}^6 (I_{N_j})^{w_j}}, \frac{\bigotimes_{j=1}^6 (1 + F_{N_j})^{w_j} - \bigotimes_{j=1}^6 (1 - F_{N_j})^{w_j}}{\bigotimes_{j=1}^6 (1 + F_{N_j})^{w_j} + \bigotimes_{j=1}^6 (1 - F_{N_j})^{w_j}} \right) \end{array} \right)$$

$$d'_1 = ([0.7231, 0.9424], [0.8123, 0.9042], [0.0167, 0.2413], 0.1325, 0.1245, 0.8952)$$

$$d'_2 = ([0.7241, 0.9523], [0.8258, 0.9752], [0.0149, 0.2253], 0.0985, 0.1436, 0.8967)$$

$$d'_3 = ([0.5862, 0.9155], [0.7012, 0.9624], [0.0337, 0.3001], 0.1325, 0.1453, 0.9157)$$

In this case $C_m(d'_1) = 0.9725$, $C_m(d'_2) = 0.9753$, and $C_m(d'_3) = 0.9574$. This yields that $y_2 > y_1 > y_3$. Here

the role of real number (base) is change by weight in neutrosophic cubic Einstein weighted aggregation operator (NCEWA) operator, we observe the we get the same result as by NCEWA operator.

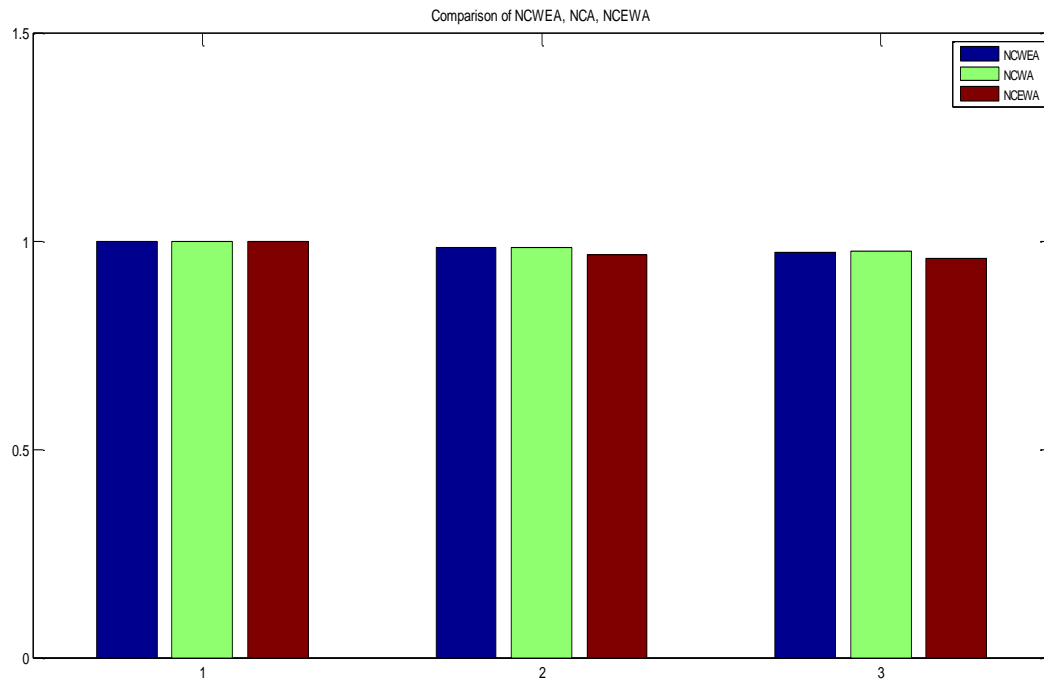


Fig.2: Graphical comparison of NCWEA with NCWA and NCEWA.

From the graph it is clear that NCWEA produce same result as NCWA and NCEWA, so we have a good tool to deal the cases in which neutrosophic cubic values appears in exponential form.

9. Conclusion

This manuscript presents a novel exponential operational laws (EOL) on NCSs with base crisp value and interval value which is an effective addition to the existing laws. We evaluated some properties and relations. Based upon these EOLs, we established NCWEA and DNCWEA operators. These aggregation operators are applied to establish a MADM for solving the daily life problem with the neutrosophic cubic information. The proposed method is used upon a daily life problem as an application. A comparative analysis with neutrosophic cubic weighted aggregation operator (NCWA) and neutrosophic cubic Einstein weighted aggregation operator (NCEWA) is provided to show the effectiveness of the approach. The graphical representation is accomplished between these operators. It is concluded that these newly defined operational laws and the proposed aggregation operators can parallelly be used to solve the MADM problems in more accomplished manner.

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On Some Special Substructures of Neutrosophic Rings and Their Properties

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Abstract

In this paper we introduce the notions of AH-ideal and AHS-ideal as new kinds of neutrosophic substructures defined in a neutrosophic ring. We investigate the properties of these substructures and some related concepts as AH-weak principal ideal, AH-weak prime ideal and AH-weak maximal ideal.

Keywords: Neutrosophic ring, AH-ideal, AHS-ideal, Neutrosophic substructure, AHS-homomorphism.

1. Introduction

Neutrosophy as a branch of philosophy introduced by Smarandache has many applications in both real world and mathematical concepts, especially, in algebra. The notion of neutrosophic groups and rings is defined by Kandasamy and Smarandache in [9], and studied widely in [3, 4, 7]. Studies were carried out on neutrosophic rings, neutrosophic hyperring, and neutrosophic refined rings. See [1-2]. Neutrosophic rings have many interesting properties and substructures such as neutrosophic subrings and neutrosophic ideals. They are defined and studied widely. See [1, 3, 4]. In this work we focus on subsets with form $P+QI$ where P, Q are ideals in the ring R . Two new kinds of neutrosophic substructures which we call AH-ideals and AHS-ideals can be defined by the previous aspect. We prove many theorems which describe their essential properties. Also, we introduce some related concepts such as AH-weak principal ideal, AH-weak prime ideal and AH-weak maximal ideal which have many interesting properties similar to the properties of the classical principal, maximal and prime ideals defined in classical rings.

For our purpose we introduce the concept of AHS-homomorphism and AHS-isomorphism.

Motivation

Since the neutrosophic ring under addition and multiplication ($+$ and \times) $R(I) = \{a+bI; a, b \in R, R \text{ is a ring}\}$ can be represented as $R+RI$ [4], we are interested in studying the subsets with form $P+QI$; where P, Q are ideals in R , in addition to investigating their properties.

2. Preliminaries

In this section we introduce a short revision of some theorems and definitions about ideals and neutrosophic ideals.

Definition 2.1:[8]

Let $(R, +, \times)$ be a ring and P be an ideal of R

- (a) P is called prime if $a \times b \in P$ implies $a \in P$ or $b \in P$ for $a, b \in R$.
- (b) P is called maximal if there is no proper ideal J is containing P .
- (c) P is called principal if $P = \langle a \rangle$ for some $a \in R$.
- (d) The set $M = \{x \in R ; \exists n \in \mathbb{Z} \text{ such } x^n \in P\}$ is called the root ideal of P and we denote it by \sqrt{P} .

Theorem 2.2:[8]

Let R, T be two commutative rings and $f: R \rightarrow T$ be a ring homomorphism; let P be an ideal in R and J an ideal in T such $J \neq T$ and $\ker f \leq P \neq R$, then

- (a) P is prime in R if and only if $f(P)$ is prime in T .
- (b) P is maximal in R if and only if $f(P)$ is maximal in T .
- (c) J is prime in T if and only if $f^{-1}(J)$ is prime in R .
- (d) J is maximal in T if and only if $f^{-1}(J)$ is maximal in R .

Definition 2.3:[9]

Let $(R, +, \times)$ be a ring, then $R(I) = \{a + bI ; a, b \in R\}$ is called the neutrosophic ring; where I is a neutrosophic indeterminate element with the condition $I^2 = I$.

Definition 2.4:[9]

Let $R(I)$ be a neutrosophic ring, a non-empty subset P of $R(I)$ is called a neutrosophic ideal if :

- (a) P is a neutrosophic subring of $R(I)$.
- (b) for every $p \in P$ and $r \in R(I)$, we have $r \times p, p \times r \in P$.

Theorem 2.5: [8]

Let P, Q be two ideals in the ring R , then $P \cap Q, P + Q, P \times Q$ are ideals in R .

For definitions of $P+Q$, $P \times Q$, [see 8, pp. 49-53].

3. Main concepts and discussion

Definition 3.1:

Let R be a ring and $R(I)$ be the related neutrosophic ring and $P = P_0 + P_1I = \{a_0 + a_1I; a_0 \in P_0, a_1 \in P_1\}$; P_0, P_1 are two subsets of R .

(a) We say that P is an AH-ideal if P_0, P_1 are ideals in the ring R .

(b) We say that P is an AHS-ideal if $P_0 = P_1$.

(c) The AH-ideal P is called null if $P_0, P_1 \in \{R, O\}$.

Theorem 3.2:

Let $R(I)$ be a neutrosophic ring and $P = P_0 + P_1I$ be an AH-ideal, then P is not a neutrosophic ideal in general by the classical meaning.

Proof:

Since P_0, P_1 are ideals, they are subgroups of $(R, +)$, thus $P = P_0 + P_1I$ is a neutrosophic subgroup of $(R(I), +)$. Now suppose that

$$r = r_0 + r_1I \in R(I), a = a_0 + a_1I \in P.$$

We have $a = r_0a_0 + (r_1a_1 + r_1a_0 + r_0a_1)I$, we remark that $r_1a_1 + r_1a_0 + r_0a_1$ does not necessarily belong to P_1 because a_0 does not belong to P_1 thus P is not supposed to be an ideal. See example 3.17.

It is easy to see that if $P_0 = P_1$, then $P = P_0 + P_1I$ is a neutrosophic ideal in the classical meaning.

Remark 3.3:

We can define the right AH-ideal as P_0, P_1 are right ideals in R , and the left AH-ideal as P_0, P_1 are left ideals in R .

Definition 3.4:

Let $R(I)$ be a neutrosophic ring and $P = P_0 + P_1I$, $Q = Q_0 + Q_1I$ be two AH-ideals. Then we define:

$$P + Q = (P_0 + Q_0) + (P_1 + Q_1)I.$$

$$P \cap Q = (P_0 \cap Q_0) + (P_1 \cap Q_1)I.$$

$$P \times Q = P_0Q_0 + (P_1Q_0 + P_0Q_1 + P_1Q_1)I.$$

Theorem 3.5:

Let $R(I)$ be a neutrosophic ring and $P = P_0 + P_1I$, $Q = Q_0 + Q_1I$ be two AH-ideals, then

$P + Q$ and $P \cap Q$, $P \times Q$ are AH-ideals.

Proof:

$P_i \times Q_i$, $P_i + Q_i$, $P_i \cap Q_i$ for $i \in \{0,1\}$ are ideals in R as a result of Theorem 2.5, thus we get the proof. See example 3.17.

Definition 3.6:

Let $R(I)$ be a commutative neutrosophic ring and $P = P_0 + P_1I$ be an AH-ideal then the AH-root of P can be defined as: $AH - Rad(P) = \sqrt{P_0} + \sqrt{P_1}I$.

Theorem 3.7:

Every AH-root of an AH-ideal is also AH-ideal.

Proof:

Since $\sqrt{P_i}$ is an ideal in R we get that $\sqrt{P_0} + \sqrt{P_1}I$ is an AH-ideal of the neutrosophic ring $R(I)$.

It is easy to see that if P is an AHS-ideal then the AH-root of P is also an AHS-ideal because $\sqrt{P_0} = \sqrt{P_1}$.

Definition 3.8:

Let $R(I)$ be a neutrosophic ring and $P = P_0 + P_1I$ be an AH-ideal. Then we define the AH-factor as: $R(I)/P = R/P_0 + R/P_1I$.

Theorem 3.9:

Let $R(I)$ be a neutrosophic ring and $P = P_0 + P_1I$ be an AH-ideal then $R(I)/P$ is a ring with the following two binary operations

$$[(x_0 + P_0) + (y_0 + P_1)I] + [(x_1 + P_0) + (y_1 + P_1)I] =$$

$$[(x_0 + x_1 + P_0) + (y_0 + y_1 + P_1)I]$$

$$[(x_0 + P_0) + (y_0 + P_1)I] \times [(x_1 + P_0) + (y_1 + P_1)I] = [(x_0 \times x_1 + P_0) + (y_0 \times y_1 + P_1)I].$$

Proof:

Since P_0, P_1 are ideals in R , then $R/P_0, R/P_1$ are rings, so $R(I)/P$ under the previous operations is closed. It is obvious that $(R(I)/P, +)$ is abelian neutrosophic group.

Addition is well defined, suppose that $[(x_0 + P_0) + (y_0 + P_1)I] = [(x_1 + P_0) + (y_1 + P_1)I]$ so $(x_0 + P_0) = (x_1 + P_0)$ and $(y_0 + P_1) = (y_1 + P_1)$ thus $x_1 - x_0 \in P_0, y_1 - y_0 \in P_1$

$[(x_2 + P_0) + (y_2 + P_1)I] = [(x_3 + P_0) + (y_3 + P_1)I]$ so $(x_2 + P_0) = (x_3 + P_0)$ and $(y_2 + P_1) = (y_3 + P_1)$ thus $x_3 - x_2 \in P_0, y_3 - y_2 \in P_1$

$[(x_0 + P_0) + (y_0 + P_1)I] + [(x_2 + P_0) + (y_2 + P_1)I] = [(x_0 + x_2 + P_0) + (y_0 + y_2 + P_1)I]$ and

$$[(x_1 + P_0) + (y_1 + P_1)I] + [(x_3 + P_0) + (y_3 + P_1)I] = [(x_1 + x_3 + P_0) + (y_1 + y_3 + P_1)I]$$

We can see that $(x_1 + x_3) - (x_0 + x_2) = (x_1 - x_0) + (x_3 - x_2) \in P_0$ and $(y_1 + y_3) - (y_0 + y_2) = (y_1 - y_0) + (y_3 - y_2) \in P_1$ thus $[(x_1 + x_3 + P_0) + (y_1 + y_3 + P_1)I] = [(x_0 + x_2 + P_0) + (y_0 + y_2 + P_1)I]$.

Multiplication is well defined. Since $x_1 - x_0 \in P_0$ then $(x_1 - x_0) \times x_2 = x_1 \times x_2 - x_0 \times x_2 \in P_0$, by the same we find $x_1 \times (x_3 - x_2) = x_1 \times x_3 - x_1 \times x_2 \in P_0$ that implies $(x_1 \times x_2 - x_0 \times x_2) + (x_1 \times x_3 - x_1 \times x_2) = (x_1 \times x_3 - x_0 \times x_2) \in P_0$

By the same argument we find $(y_1 \times y_3 - y_0 \times y_2) \in P_1$ thus $(y_1 \times y_3 + P_1) = (y_0 \times y_2 + P_1)$ and $(x_1 \times x_3 + P_0) = (x_0 \times x_2 + P_0)$; thus addition and multiplication are well defined.

The multiplication is associative and distributive with respect to addition.

Let $x = (x_0 + P_0) + (y_0 + P_1)I, y = (x_1 + P_0) + (y_1 + P_1)I, z = (x_2 + P_0) + (y_2 + P_1)I$ be three elements in $R(I)/P$ we have:

$$x \times (y + z) = [(x_0 + P_0) + (y_0 + P_1)I] \times [(x_1 + x_2 + P_0) + (y_1 + y_2 + P_1)I] =$$

$$[x_0 \times (x_1 + x_2) + P_0] + [y_0 \times (y_1 + y_2) + P_1]I = [x_0 \times x_1 + x_0 \times x_2 + P_0] + [y_0 \times y_1 + y_0 \times y_2 + P_1]I =$$

$$[(x_0 + P_0) + (y_0 + P_1)I] \times [(x_1 + P_0) + (y_1 + P_1)I] + [(x_0 + P_0) + (y_0 + P_1)I] \times [(x_2 + P_0) + (y_2 + P_1)I] = x \times y + x \times z.$$

Following the same argument, we can prove that $(y + z) \times x = y \times x + z \times x$.

Thus we get the proof.

Definiton 3.10:

Let $R(I), T(J)$ be two neutrosophic rings and the map $f: R(I) \rightarrow T(J)$ we say that f is a neutrosophic AHS-homomorphism if

The restriction of the map f on R is a ring homomorphism from R to T i.e. $f_R: R \rightarrow T$ is homomorphism and

$$f(a + bI) = f_R(a) + f_R(b)J.$$

We say that $R(I)$, $T(J)$ are AHS-isomomorphic neutrosophic rings if there is a neutrosophic AHS-homomorphism

$f: R(I) \rightarrow T(J)$ which is a bijective map i.e. ($R \cong T$), we say that f is a neutrosophic AHS-isomorphism.

Example 3.11:

Suppose that $R = (Z_6, +, \times)$, $T = (Z_{10}, +, \times)$ are two rings, we have $f: R(I) \rightarrow T(J)$; $f(a + bI) = 5a + 5bJ$ is an AHS-homomorphism because $f_R: R \rightarrow T$; $f_R(a) = 5a$ is a homomorphism between R and T .

The previous example shows that AHS-homomorphism is not supposed to be a neutrosophic ring homomorphism defined in [3] because $f(I) = f(0 + 1.I) = f(0) + f(1)J = 0 + 5J = 5J \neq J$.

It is easy to see that if $f: R(I) \rightarrow T(J)$ is a neutrosophic AHS-homomorphism then $f(R(I)) = f_R(R) + f_R(R)J$.

The AH-kernel of $f: R(I) \rightarrow T(J)$ can be defined as $AH - Kerf = Kerf_R + Kerf_R J$

In the last example we have $Kerf_R = \{0, 2, 4\}$ thus $AH - kerf = Kerf_R + Kerf_R I = \{0, 2, 4, 2I, 4I, 2+4I, 2+2I, 4+2I, 4+4I\}$.

If $Q = Q_0 + Q_1J$ is an AH-ideal of $T(J)$, then the inverse image of Q is

$$f^{-1}(Q) = f_T^{-1}(Q_0) + f_T^{-1}(Q_1)I.$$

Theorem 3.12:

Let $R(I)$, $T(J)$ be two neutrosophic rings and $f: R(I) \rightarrow T(J)$ is a neutrosophic ring AHS-homomorphism, let $P = P_0 + P_1I$ be an AH-ideal of $R(I)$ and $Q = Q_0 + Q_1J$ be an AH-ideal of $T(J)$, then we have

- (a) $f(P)$ is an AH-ideal of $f(R(I))$.
- (b) $f^{-1}(Q)$ is an AH-ideal of $R(I)$.
- (c) If P is AHS-ideal of $R(I)$, then $f(P)$ is an AHS-ideal of $f(R(I))$.
- (d) $AH - kerf = kerf_R + kerf_R I$ is an AHS-ideal; f_R is the restriction of f on the ring R .
- (e) The AH-factor $R(I)/kerf$ is AHS - isomorphic to $f(R(I))$.

Proof:

- (a) Since f can be restricted on R , by Definition 3.10, we can write

$f(P) = f_R(P_0) + f_R(P_1)J$. Since $f_R(P_i); i \in \{0,1\}$ is an ideal in $f(R)$, thus $f(P)$ is an AH-ideal in $f(R(I))$.

(b) Since $f^{-1}(Q) = f_T^{-1}(Q_0) + f_T^{-1}(Q_1)I$ and $f_T^{-1}(Q_i); i \in \{0,1\}$ is an ideal in R so $f^{-1}(Q)$ is an AH-ideal of $T(J)$.

(c) We have $P_0 = P_1$, so $f_R(P_0) = f_R(P_1)$ and $f(P)$ must be an AHS-ideal.

(d) Since $\ker f_R$ is an ideal of R then $AH - \ker f = \ker f_R + \ker f_R I$ is an AHS-ideal of $R(I)$.

(e) Since f is a ring homomorphism, then $R/\ker f_R \cong f(R)$ so we get:

$$R(I)/\ker f = R/\ker f_R + R/\ker f_R J \cong f_R(R) + f_R(R)J = f(R(I)).$$

We mean by the symbol \cong the concept of AHS-isomorphism introduced in Definition 3.10.

[For more clarity see Examples 3.17 and 3.18].

Definition 3.13:

Let $R(I)$ be a neutrosophic commutative ring and $P = P_0 + P_1 I$ be an AH-ideal. Then we say that

- (a) P is a weak prime AH-ideal if P_0, P_1 are prime ideals in R .
- (b) P is a weak maximal AH-ideal if P_0, P_1 are maximal ideals in R .
- (c) P is a weak principal AH-ideal if P_0, P_1 are principal ideals in R .

Definition 3.14:

Let $R(I)$ be a commutative neutrosophic ring, we call it a weak principal AH-ring if every AH-ideal is a weak AH-principal ideal.

Theorem 3.15:

Let $R(I), T(J)$ be two commutative neutrosophic rings with a neutrosophic AHS-homomorphism $f: R(I) \rightarrow T(J)$ then

If $P = P_0 + P_1 I$ is an AHS-ideal of $R(I)$ and $AH - \ker f \leq P \neq R(I)$ then

- (a) P is a weak prime AHS-ideal if and only if $f(P)$ is a weak prime AHS-ideal in $f(R(I))$.
- (b) P is a weak maximal AHS-ideal if and only if $f(P)$ is a weak maximal AHS-ideal in $f(R(I))$.
- (c) If $Q = Q_0 + Q_1 J$ is an AH-ideal of $T(J)$ then it is a weak prime AH-ideal if and only if $f^{-1}(Q)$ is a weak prime in $R(I)$.
- (d) If $Q = Q_0 + Q_1 J$ is an AHS-ideal of $T(J)$ then it is a weak maximal AHS-ideal if and only if $f^{-1}(Q)$ is a weak maximal in $R(I)$.

Proof:

(a) We have $\text{AH-Ker } f \leq P$ so $\ker f_R \leq P_0, \ker f_R \leq P_1$. We can find

$f_R(P_0) = f_R(P_1)$ and both of them are ideals in $f(R)$; thus $f(P) = f_R(P_0) + f_R(P_1)I$ is a weak prime AHS-ideal in $f(R(I))$ if and only if P is a weak prime AH-ideal in $R(I)$ as a result of theorem 2.2.

(b) Following the same argument, we can get the proof.

(c) We have that $f^{-1}(Q) = f_T^{-1}(Q_0) + f_T^{-1}(Q_1)I$, and $f_T^{-1}(Q_0), f_T^{-1}(Q_1)$ are prime in R if and only if Q_0, Q_1 are prime in T , then the proof holds.

(d) We have $f^{-1}(Q) = f_T^{-1}(Q_0) + f_T^{-1}(Q_1)I$, and $f_T^{-1}(Q_0), f_T^{-1}(Q_1)$ are maximal in R if and only if Q_0, Q_1 are maximal in T by theorem (2.2), thus the proof holds.

Remark 3.16:

It is easy to see that if P is an AH-ideal in $R(I)$ then (a) and (b) are still true.

(c), (d) are still true if Q is an AHS-ideal.

Example 3.17:

In this example we clarify some of introduced concepts.

Let $R(I) = Z_6(I)$, $P_0 = \{0, 2, 4\}$, $P_1 = \{0, 3\}$ are two ideals in Z_6 then we have

(a) $P = P_0 + P_1I = \{0, 2, 4, 2+3I, 4+3I, 3I\}$ is an AH-ideal.

(b) $Q = P_1 + P_1I = \{0, 3, 3 + 3I, 3I\}$ is an AHS-ideal because $P_1 = P_1$.

(c) We have: $R/P_0 = \{P_0, 1+P_0\}$ and $R/P_1 = \{P_1, 1+P_1, 2+P_1\}$; thus the AH-factor

$$R(I)/P = \{P_0 + P_1I, P_0 + (1+P_1)I, P_0 + (2+P_1)I, (1+P_0) + P_1I, (1+P_0) + (1+P_1)I, (1+P_0) + (2+P_1)I\}$$

We should remark that $P_0 = P_0 + 0.I$ and $0 = 0 + 0.I$.

(d) We can clarify the addition on the AH-factor $R(I)/P$ as:

$$[P_0 + (1+P_1)I] + [(1+P_0) + (2+P_1)I] = [(0+1)+P_0] + [(1+2)+P_1]I = (1+P_0) + (3+P_1)I = (1+P_0) + P_1I.$$

We can clarify the multiplication on the AH-factor $R(I)/P$ as:

$$[P_0 + (1+P_1)I] \times [(1+P_0) + (2+P_1)I] = [(0 \times 1) + P_0] + [(1 \times 2) + P_1]I = P_0 + (2+P_1)I.$$

(e) We can see that $P \cap Q = (P_0 \cap P_1) + (P_1 \cap P_1)I = \{0\} + P_1I = \{0, 3I\}$ which it is an AH-ideal.

$$(f) P+Q = (P_0 + P_1) + (P_1 + P_1)I = R + P_1I = \{0,1,2,3,4,5,3I, 1 + 3I, 2 + 3I, \dots 5 + 3I\}.$$

Example 3.18:

Suppose that $R = (Z_6, +, \times), T = (Z_{10}, +, \times)$ are two commutative rings, we have $f: R(I) \rightarrow T(J); f(a + bI) = 5a + 5bJ$.

f is a neutrosophic AHS-homomorphism because $f_R: R \rightarrow T; f_R(a) = 5a$ is a homomorphism.

We have: $P = \text{Ker} f_R = \{0,2,4\}, f_R(R) = \{0,5\}, R/P \cong f_R(R) = \{0,5\}, R/P = \{P, (1+P)\}$.

The AH-factor $R(I)/\text{Ker} f = R/P + R/P J = \{(P + P.J), (P + [1+P]J), ([1+P]+P J), ([1+P]+[1+P]J)\}$

Which is AH-isomorphic to $f(R(I)) = f_R(R) + f_R(R)J = \{0,5\} + (\{0,5\})J = \{0, 5, 5J, 5 + 5J\}$.

$Q = P_1 + P_1I = \{0,3,3 + 3I, 3I\}$ is an AHS-ideal defined in Example 3.17, we have $f(Q) = \{0,5,5J, 5 + 5J\}$, which is an AHS-ideal of $T(J)$.

$S_0 = \{0,2,4,6,8\}$ is a ideal of T thus $S = S_0 + S_0J = \{0,2,4,6,8,2J, 4J, 6J, 8J, 2 + 2J, 2 + 4J, 2 + 6J, 2 + 8J, 4 + 2J, 4 + 4J, 4 + 6J, 4 + 8J, 6 + 2J, 6 + 4J, 6 + 6J, 6 + 8J\}$ is an AHS-ideal of $T(J)$.

$f_T^{-1}(S_0) = \{0,2,4\}$ so $f^{-1}(S) = f_T^{-1}(S_0) + f_T^{-1}(S_0)I = \{0,2,4,2I, 4I, 2 + 2I, 2 + 4I, 4 + 2I, 4 + 4I\}$ is an AHS-ideal of $R(I)$.

Example 3.19:

In the ring $R = (Z_6, +, \times)$ we have two maximal ideals $P = \{0,3\}, Q = \{0,2,4\}$ thus $P+QI$ and $Q+PI$ are two weak maximal AH-ideals of $R(I)$.

Example 3.20:

(a) In the ring $R = (Z_8, +, \times)$ we have only one maximal ideal $P = \{0,2,4,6\}$ so $P+PI$ is a weak maximal AHS-ideal of $R(I)$.

(b) We have $Q = \{0,4\}$ is an ideal in $R, \sqrt{Q} = \{0,2,4,6\} = P$ thus the AH-root of $Q+QI$ is equal to $P+PI$.

Example 3.21:

In the ring $(Z, +, \times)$ each ideal P is principal thus each AH-ideal $S = P+QI$ is weak principal AH-ideal so $Z(I)$ is a weak principal AH-ring.

Example 3.22:

(a) In the ring $(Z, +, \times)$, $P = \langle 3 \rangle, Q = \langle 2 \rangle$ are two prime and maximal ideals so $P+QI = \{3n+2mI; n, m \in Z\}$ is weak prime AH-ideal and weak maximal AH-ideal.

(b) The map $f_Z: Z \rightarrow Z_6; f(a) = a \bmod 6$ is a homomorphism so the related AH-homomorphism is

$f: Z(I) \rightarrow Z_6(J); f(a + bI) = [a \bmod 6] + [b \bmod 6]J$ and $\text{AH-kerf} = 6Z + 6ZI$ is contained in $P + QI$.

(c) $f(P + QI) = f(P) + f(Q)J = \{0, 3\} + \{0, 2, 4\}J$ which is a weak maximal / prime AH-ideal in $Z_6(I)$ since $\{0, 3\}$, $\{0, 2, 4\}$ are maximal and prime in Z_6 .

(d) Since $Q = \{0, 2, 4\}$ is maximal in Z_6 , $P = Q + QJ$ is a weak maximal / prime AH-ideal of $Z_6(J)$ and we find

$f^{-1}(P) = f_{Z_6}^{-1}(Q) + f_{Z_6}^{-1}(Q)I = \langle 2 \rangle + \langle 2 \rangle I$ which is a weak maximal / prime AHS-ideal in $Z(I)$.

Conclusion

In this article we introduced the concepts of AH-ideals and AHS-ideals in a neutrosophic ring. Some related concepts as weak principal ideal, AH-weak prime ideal and AH-weak maximal ideal are presented with some useful tools as AHS-homomorphism/isomorphism. We investigated the essential properties of these concepts and proved many related theorems concerning these properties.

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A new ranking function of triangular neutrosophic number and its application in integer programming

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Abstract

Real humankind problems have different sorts of ambiguity in the creation, and amidst them, one of the significant issues in solving the integer linear programming issues. In this commitment, the conception of aggregation of ranking function has been focused on a distinct framework of reference. Here, we build up another framework for neutrosophic integer programming issues having triangular neutrosophic numbers by using the aggregate ranking function. To legitimize the proposed technique, scarcely numerical analyses are given to show the viability of the new model. At long last, conclusions are talked about.

Keywords: Neutrosophic triangular numbers, integer programming, aggregate ranking function.

1.Introduction

Professor Zadeh [1] was originally presented the idea of a fuzzy set theory (in 1965). The idea of fuzziness has a leading feature to solve efficiently in engineering and statistical problem. Applying the uncertainty theory, plentiful varieties of realistic problems can be solved, networking problems, decision-making problems, influence on social science, etc. As a result, by considering fuzzy parameters in linear programming, fuzzy linear programming is defined. Accordingly, various researchers have demonstrated their attentiveness to various sorts of fuzzy linear programming (FLP) issue and proposed a diverse system for dealing with FLP issues. If the parameters and constraints are fuzzy numbers, then it is called fully fuzzy numbers. A general class of fully FLP (FFLP) was introduced by Buckley and Featuring [46]. Many authors [2, 30-35, 37] considered issues either fuzzy linear programming implies either just the right-hand side or the constraints have been fuzzy or simply factors are fuzzy. Fuzzy IP problem is also the main part of LP problem. Allahviranloo et al. [3] offered a technique for solving IP problems. Fan et al., also [4] offered a general technique for resolving IP under fuzzy environment. Dehghan et al. [38] proposed practical methodologies to resolve a fully fuzzy linear system (FFLS) that is proportional to the remarkable systems. Lotfi et al. [39] proposed a procedure for symmetric triangular fuzzy number, gained another system for dealing with FFLP issues by changing over two relating LPs. To overcome these obstruction Kumar et al. [36] introduced another system for discovering the fuzzy ideal arrangement of FFLP issue with uniformity imperatives. After that Edalatpanah [14-15], Das [8-12], Das

et al. [5-6] have portrayed to deal with the FFLP issue with the assistance of situating limit and lexicographic methodology. Wan and Dong [42] proposed a new approach for trapezoidal fuzzy linear programming problems thinking about the acceptance degree of fuzzy constraints violate.

Pertaining the concept of Zadeh's research paper, Atanassov [40] created phenomenally the intuitionistic fuzzy set where he meticulously elucidates the concept of membership and nonmembership function. Smarandache [16] in 1998 germinated the notion of having neutrosophic set holding three different fundamental elements (i) truth, (ii) indeterminate, and (iii) falsity. Each and every attribute of the neutrosophic sets are very relevant factors to our real-life models. Afterward, Wang et al. [44] progressed with a single typed neutrosophic set, which serves the solution to any sort of complicated problem in a very efficient way. Neutrosophic hypothesis applied in numerous fields of sciences, so as to take care of the issues identified with indeterminacy, see [20, 22, 27-28, 41, 45, 48-49]. In like manner, Abdel-Baset [23] added the neutrosophic LP models the place their parameters are tended with trapezoidal neutrosophic numbers and introduced a method for getting them. Das and Jatindra [43] introduced a strategy for solving neutrosophic LP problem having triangular neutrosophic numbers by using ranking function. Edalatpanah [13, 21] presented some direct approaches of neutrosophic LP problem having the triangular fuzzy number. Again, Edalatpanah [17-20] established some aggregate ranking functions for data envelopment analysis (DEA) based on triangular neutrosophic numbers. Mohamed et al., [47] introduced another score function for neutrosophic integer programming problems having triangular neutrosophic numbers. Banerjee and Pramanik [25] added the LP problem with single objective in neutrosophic number (NN) condition with the assistance of goal programming. Likewise, Pramanik and Dey [24] detailed arrangement technique to linear bi-level programming problem in NN condition. Maiti et al. [26] introduced a strategy for multi-level-multi-objective LP problems by the assistance of goal programming. Hussian et al. [29] proposed a neutrosophic LP issue using ranking function. A IP issue under neutrosophic condition having triangular neutrosophic numbers was proposed by Nafei and Nasser [7].

The motivation of this research paper, to develop an aggregate ranking function and usage of our function in integer programming (IP) problem. We propose IP problem based on triangular neutrosophic numbers. We likewise change the neutrosophic IP issue into a crisp IP model through the use of the aggregate ranking function. Any standard methodologies explain this crisp IP issue.

This research paper is prepared as follows: in the next segment, some fundamental concepts, mathematical operation on triangular neutrosophic numbers are introduced. In the next following segment, the proposed strategy for solving the IP problem is examined. Following this segment, the sub-section of Limitation and shortcoming of the existing method, sub-section of neutrosophic IP is discussed. In the next subsection, we discuss the proposed algorithm for solving our problem. In segment before determination, a numerical model is given to uncover the viability of the proposed model. At long last, conclusions are given in the last segment.

2. Preliminaries

Right now, some fundamental concepts and neutrosophic numbers have been examined under this segment.

Definition 1. [16]

Assume S be a space of objectives and $s \in S$. A neutrosophic set N in S may be interpret via three membership functions for truth, indeterminacy along with falsity and represent by $\rho_t(s)$, $\beta_t(s)$ and $\lambda_t(s)$ are real standard or

real nonstandard subsets of $]0^-, 1^+[$. That is $\rho_I(s) : S \rightarrow]0^-, 1^+[$, $\beta_I(s) : S \rightarrow]0^-, 1^+[$ and $\lambda_I(s) : S \rightarrow]0^-, 1^+[$. There is no limitation on the sum of $\rho_I(s)$, $\beta_I(s)$ and $\lambda_I(s)$, so $0^- \leq \sup \rho_I(s) + \sup \beta_I(s) + \sup \lambda_I(s) \leq 3^+$.

Definition 2. [16]

A single-valued neutrosophic set (SVNS) I through S taking the form $I = \{s, \rho_I(s), \beta_I(s), \lambda_I(s); s \in S\}$, where S be a space of discourse, $\rho_I(s) : S \rightarrow]0^-, 1^+[$, $\beta_I(s) : S \rightarrow]0^-, 1^+[$ and $\lambda_I(s) : S \rightarrow]0^-, 1^+[$ with $0 < \rho_I(s) + \beta_I(s) + \lambda_I(s) < 3$ for all $s \in S$. $\rho_I(s)$, $\beta_I(s)$ and $\lambda_I(s)$ respectively represent truth membership, indeterminacy membership, and falsity membership degree of s to I .

Definition3 [43]. A triangular neutrosophic number (TNNs) is signified via $I = \langle (b^1, b^2, b^3), (\alpha, \delta, \lambda) \rangle$ is an extended version of the three membership functions for the truth, indeterminacy, and falsity of s can be defined as follows:

$$\rho_I(s) = \begin{cases} \frac{(s - b^1)}{(b^2 - b^1)} \alpha & b^1 \leq s < b^2, \\ \alpha & s = b^1, \\ \frac{(b^3 - s)}{(b^3 - b^2)} \alpha & b^2 \leq s < b^3, \\ 0 & \text{something else.} \end{cases}$$

$$\beta_I(s) = \begin{cases} \frac{(b^2 - s)}{(b^2 - b^1)} \delta, & b^1 \leq s < b^2, \\ \delta, & s = b^2, \\ \frac{(s - b^3)}{(b^3 - b^1)} \delta, & b^2 \leq s < b^3, \\ 1, & \text{something else.} \end{cases}$$

$$\lambda_I(s) = \begin{cases} \frac{(b^1 - s)}{(b^2 - b^1)} \lambda, & b^1 \leq s < b^2, \\ \lambda, & s = b^2, \\ \frac{(s - b^3)}{(b^3 - b^2)} \lambda, & b^2 \leq s < b^3, \\ 1, & \text{something else.} \end{cases}$$

Where, $0 \leq \rho_I(s) + \beta_I(s) + \lambda_I(s) \leq 3, s \in I$. Additionally, when $b^1 \geq 0$, I is called a nonnegative TNN.

Similarly, when $b^1 < 0$, I becomes a negative TNN.

Definition 5 [19]. Arithmetic Operation

Suppose $A_1^I = \langle (b_1^1, b_1^2, b_1^3), (\alpha_1, \delta_1, \lambda_1) \rangle$ and $A_2^I = \langle (b_2^1, b_2^2, b_2^3), (\alpha_2, \delta_2, \lambda_2) \rangle$ be two TNNs. Then the mathematical computation will be explained as:

$$(i) A_1^I \oplus A_2^I = \langle (b_1^1 + b_2^1, b_1^2 + b_2^2, b_1^3 + b_2^3), (\alpha_1 \wedge \alpha_2, \delta_1 \vee \delta_2, \lambda_1 \vee \lambda_2) \rangle$$

$$(ii) A_1^M - A_2^M = \langle (b_1^1 - b_2^1, b_1^2 - b_2^2, b_1^3 - b_2^3), (\alpha_1 \wedge \alpha_2, \delta_1 \vee \delta_2, \lambda_1 \vee \lambda_2) \rangle$$

$$(iii) A_1^M \otimes A_2^M = \langle (b_1^1 b_2^1, b_1^2 b_2^2, b_1^3 b_2^3), (\alpha_1 \wedge \alpha_2, \delta_1 \vee \delta_2, \lambda_1 \vee \lambda_2) \rangle, \text{ if } b_1^1 > 0, b_2^1 > 0,$$

$$(iv) \lambda A_1^M = \begin{cases} \langle (\lambda b_1^1, \lambda b_1^2, \lambda b_1^3), (\alpha_1, \delta_1, \lambda_1) \rangle, & \text{if } \lambda > 0 \\ \langle (\lambda b_1^3, \lambda b_1^2, \lambda b_1^1), (\alpha_1, \delta_1, \lambda_1) \rangle, & \text{if } \lambda < 0 \end{cases}$$

$$(v) \frac{A_1^M}{A_2^M} = \begin{cases} \left\langle \left(\frac{b_1^1}{b_2^3}, \frac{b_1^2}{b_2^2}, \frac{b_1^3}{b_2^1} \right); \alpha_1 \wedge \alpha_2, \delta_1 \wedge \delta_2, \lambda_1 \wedge \lambda_2 \right\rangle & (b_2^1 > 0, b_2^3 > 0) \\ \left\langle \left(\frac{b_1^3}{b_2^3}, \frac{b_1^2}{b_2^2}, \frac{b_1^1}{b_2^1} \right); \alpha_1 \wedge \alpha_2, \delta_1 \wedge \delta_2, \lambda_1 \wedge \lambda_2 \right\rangle & (b_2^1 < 0, b_2^3 > 0) \\ \left\langle \left(\frac{b_1^3}{b_2^1}, \frac{b_1^2}{b_2^2}, \frac{b_1^1}{b_2^3} \right); \alpha_1 \wedge \alpha_2, \delta_1 \wedge \delta_2, \lambda_1 \wedge \lambda_2 \right\rangle & (b_2^1 < 0, b_2^3 < 0) \end{cases}$$

3. Proposed model

Before going to our main algorithm, first of all, we tend to begin a subsidiary i.e., drawback as well as restriction of the available method [7]

3.1 Shortcoming and Limitation of the existing method

First of all, we investigate drawback as well as restrictions of an available method [7] under exclusive ranking function.

Nafei and Nasser [7] suggested a model for IP problems by utilizing the ranking function. However, the author uses some scientific presumption to resolve the problem that may be invalid in any case. This has been examined in Example 3.1 and Example 3.2.

Definition 6: One can examine any two TNNs in response to the ranking functions. Let

$I^N = \langle (b^1, b^2, b^3); \alpha, \delta, \lambda \rangle$ be a triangular neutrosophic numbers (TNNs); then

$$R(I^N) = \frac{b^1 + b^3 + 2b^2}{4} + |\alpha - \delta - \lambda|$$

Example-3.1 Let $I^N = \langle (4, 8, 10); 0.5, 0.3, 0.6 \rangle$ and $I_1^N = \langle (3, 7, 11); 0.4, 0.5, 0.6 \rangle$ then $R(I^N) = 7.9$ and $R(I_1^N) = 7.7$

Based on definition 5:

$$I^N + I_1^N = \langle (7, 15, 21); 0.4, 0.5, 0.6 \rangle \text{ then } R(I^N + I_1^N) = 15.2$$

We observe that $R(I^N + I_1^N) \neq R(I^N) + R(I_1^N)$.

Here, we observed that the author used an invalid mathematical assumption i.e. ranking function, to solve the problem. Therefore, we consider an aggregation ranking function, which was defined by Edalatpanah [19].

Definition 7. Let $I^N = \langle (b^1, b^2, b^3); \alpha, \delta, \lambda \rangle$ be a triangular neutrosophic numbers (TNNs); then the aggregation ranking function is as follows:

$$R(I^N) = \frac{(2 + \min \alpha - \max \delta - \max \lambda)}{9} \sum (b^1 + b^2 + b^3)$$

Example-3.2 Let $I^N = \langle (4, 8, 10); 0.5, 0.3, 0.6 \rangle$ and $I_1^N = \langle (3, 7, 11); 0.4, 0.5, 0.6 \rangle$ then $R(I^N) = 3.91$ and $R(I_1^N) = 3.03$

Based on definition 5:

$$I^N + I_1^N = \langle (7, 15, 21); 0.4, 0.5, 0.6 \rangle \text{ then } R(I^N + I_1^N) = \frac{(2 + 0.4 - 0.5 - 0.6)}{9} \times (22 + 21) = 6.94$$

Hence,

$$R(I^N + I_1^N) = R(I^N) + R(I_1^N).$$

Here, we observed from the above examples the existing method [7] uses the ranking function is invalid, and Definition-7 of ranking function is valid. Therefore, we consider the aggregation ranking function to solve integer programming.

3.2 Neutrosophic IP model

In this section, IP problem with neutrosophic elements are often described as the succeeding :

$$\text{Max } Z' = \sum_{j=1}^n \tilde{c}_j x_j$$

Subject to (1)

$$\sum_{j=1}^n \tilde{a}_{ij} x_j \leq \tilde{b}_i, \quad i = 1, 2, \dots, m,$$

$$x_j \geq 0, j = 1, 2, \dots, n. \text{ and it is an integer}$$

where x_j is nonnegative neutrosophic triangular numbers and $\tilde{c}_j, \tilde{a}_{ij}, \tilde{b}_i$ represented the neutrosophic numbers.

In our neutrosophic model we choose to maximize the degree of acceptance and limit the degree of rejection and indeterminacy of the neutrosophic objective function and constraints. Now the model are often typed as follows:

$$\begin{aligned} \max & \rho(x) \\ \min & \beta(x) \\ \min & \lambda(x) \end{aligned}$$

Subject to

$$\begin{aligned} \rho(x) &\geq \lambda(x) \\ \rho(x) &\geq \beta(x) \\ 0 \leq \rho(x) + \beta(x) + \lambda(x) &\leq 3 \\ \rho(x), \beta(x), \lambda(x) &\geq 0 \\ x &\geq 0 \text{ is integer.} \end{aligned} \tag{2}$$

The problem may be typed for the equal structure as follows:

$$\max \alpha, \min \delta, \min \lambda$$

Subject to

$$\begin{aligned} \alpha &\leq \rho(x) \\ \lambda &\geq \beta(x) \\ \delta &\geq \lambda(x) \\ \alpha &\geq \lambda \\ \alpha &\geq \delta \\ 0 \leq \alpha + \delta + \lambda &\leq 3 \\ x &\geq 0. \end{aligned} \tag{3}$$

the place α represents the least degree of acceptance, δ represents the largest degree of rejection and λ represents the largest degree of indeterminacy.

Now the model may be turned into the following model:

$$\max(\alpha - \delta - \lambda)$$

Subject to

$$\begin{aligned} \alpha &\leq \rho(x) \\ \lambda &\geq \beta(x) \\ \delta &\geq \lambda(x) \\ \alpha &\geq \lambda \\ \alpha &\geq \delta \\ 0 \leq \alpha + \delta + \lambda &\leq 3 \\ x &\geq 0, \text{ is integer.} \end{aligned} \tag{4}$$

Finally, the model may be typed as:

$$\min(1 - \alpha) + \delta + \lambda$$

Subject to

$$\begin{aligned} \alpha &\leq \rho(x) \\ \lambda &\geq \beta(x) \\ \delta &\geq \lambda(x) \\ \alpha &\geq \lambda \\ \alpha &\geq \delta \\ 0 &\leq \alpha + \delta + \lambda \leq 3 \\ x &\geq 0, \text{ is integer.} \end{aligned} \quad (5)$$

4. Proposed method

Here, we propose an algorithm which is solved our problem (1) and the steps are given as:

Step 1. Build the problem as the model (1).

Step 2. Consider $\tilde{b} = \langle b^l, b^m, b^r; T_b, I_b, F_b \rangle, \tilde{c} = \langle c^l, c^m, c^r; T_c, I_c, F_c \rangle$ and using Definition 5, the LP problem (1) can be transformed into problem (6).

$$\begin{aligned} \text{Max (or Min)} Z &= \sum_{j=1}^n (c_j^l, c_j^m, c_j^r) x_j \\ \text{subject to} \end{aligned} \quad (6)$$

$$\sum (a_{ij1}, a_{ij2}, a_{ij3}; T_a, I_a, F_a) x_j \leq (b_i^l, b_i^m, b_i^r; T_b, I_b, F_b), i = 1, 2, \dots, m.$$

$$x_j \geq 0, \text{ integer}, j = 1, 2, \dots, n.$$

Step 3. Using arithmetic operations, defined in Section 2 and Definition 7, the problem obtained in Step-2, is converted into the following crisp IP problem.

$$\begin{aligned} \text{Max (or Min)} Z &= \Re \sum_{j=1}^n (c_j^l, c_j^m, c_j^r) x_j \\ \text{subject to} \end{aligned} \quad (7)$$

$$\Re \sum (a_{ij1}, a_{ij2}, a_{ij3}; T_a, I_a, F_a) x_j \leq \Re (b_i^l, b_i^m, b_i^r; T_b, I_b, F_b), i = 1, 2, \dots, m.$$

$$x_j \geq 0, \text{ integer}, j = 1, 2, \dots, n.$$

Step 4. Find the optimal solution x_j by solving the crisp IP problem got in Step-3.

Step 5. Find the optimal value by placing x_j in $\sum_{j=1}^n \tilde{c}_j x_j$.

Step 6. From Step 5, solve the LP problem using the simplex method ignoring integer restrictions. If the obtained solution satisfies integer restrictions, then Stop, otherwise go to the next step by using Gomory's cutting plane algorithm.

Step 7. Add the constraints to the given set of constraints of the problem and solve the modified problem. If its optimal solution is integral, stop, otherwise repeating the step till an optimal integer solution is obtained.

5. Numerical Example

Here, we consider a case of [7] to represent the model and to quantify the effectiveness of our proposed model, we tackle a numerical model.

Example-1

$$\begin{aligned} & \max \tilde{4}x_1 + \tilde{3}x_2 \\ & s.t. \\ & \quad \tilde{4}x_1 + \tilde{2}x_2 \leq \tilde{12} \\ & \quad \tilde{3}x_1 + \tilde{6}x_2 \leq \tilde{5} \\ & \quad x_1, x_2 \geq 0, \text{ integer.} \end{aligned}$$

where

$$\begin{aligned} \tilde{4} &= \langle (2, 4, 6); (0.8, 0.6, 0.4) \rangle \\ \tilde{3} &= \langle (1, 3, 5); (0.75, 0.5, 0.3) \rangle \\ \tilde{4} &= \langle (0, 4, 8); (1, 0.0, 0.5) \rangle \\ \tilde{2} &= \langle (1, 2, 3); (1, 0.5, 0.5) \rangle \\ \tilde{12} &= \langle (5, 12, 19); (1, 0.25, 0.25) \rangle \\ \tilde{3} &= \langle (1, 3, 5); (0.75, 0.0, 0.25) \rangle \\ \tilde{5} &= \langle (3, 5, 7); (0.8, 0.6, 0.4) \rangle \\ \tilde{6} &= \langle (1, 6, 11); (1, 0, 0) \rangle \end{aligned}$$

By utilizing the aggregation ranking function proposed in Definition 7 the above issue can be changed over to crisp model as follows:

$$\begin{aligned} & \max 2.4x_1 + 1.95x_2 \\ & s.t. \\ & \quad 3.33x_1 + 1.3x_2 \leq 6 \\ & \quad 2.5x_1 + 6x_2 \leq 3 \\ & \quad x_1, x_2 \geq 0, \text{ integer} \end{aligned}$$

By following the steps introduced in the last segment, the optimal solution integer programming problem of the above problem is $x_1 = 1, x_2 = 0$ and the objective solution is $Z = 2.4$.

6. Conclusions

In this investigation, we present the neutrosophic IP and suggest a novel model to tackle it. In view of the present ranking function of triangular neutrosophic, numbers are not valid, therefore a aggregation ranking function was adopted for solving the problem. A new algorithm, the use of these ranking functions, is introduced to gain the effectivity of IP problems. For calculating the integer programming, we use Gomory's cutting plane algorithm. At long last, we utilize a numerical application to delineate the common sense and legitimacy of the proposed strategy. Also, the weaknesses of the current calculations are brought up and to show the benefits of the proposed calculations. At long last, from the acquired outcomes, it tends to be presumed that the model is proficient and advantageous.

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A STUDY OF NEUTROSOPHIC CUBIC MN SUBALGEBRA

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Abstract

In this paper, we present the new kind of MN-subalgebra for neutrosophic cubic set which is called neutrosophic cubic MN-subalgebra where M represents the initial of author's first name Mohsin and N represents the initial of second author's first name Neha. We investigate this neutrosophic cubic MN-subalgebra on BF-algebra through some significant properties of BF-algebra. We also use R-intersection, p-intersection, p-union upper bound, lower bound and some important characteristics to study the behaviour of neutrosophic cubic MN-subalgebra [NCMNSU] on BF-algebra.

Keywords: BF-algebra, Neutrosophic cubic set, Neutrosophic cubic MN-subalgebra.

1. Introduction

Fuzzy and interval-valued fuzzy sets were presented by Zadeh [20,21] Jun et al. [3,4] defined the cubic set and proved the axioms for cubic subgroups. Neggers and Kim [7] defined and investigated the B-algebra. Ahn and Ko [9] studied the structure of BF-algebra. Walendziak [19] proved the conditions of B-algebra. Senapati et al. [13] worked on fuzzy dot subalgebra and interval-valued fuzzy subalgebra with respect to t-norm in B-algebra. [14,6] many researches worked on B-algebra and BG-algebra. Khalid et al. [15] studied the neutrosophic soft cubic subalgebra with different characteristics. Khalid et al. [16] studied the effects of magnification of translation for MJB-neutrosophic set. Khalid et al. [17] investigated the T-ideal under the MJB-neutrosophic on B-algebra. Khalid et al. [18] presented the multiplication of neutrosophic cubic set. Smarandache [11,12] is the first person who presented the theory of neutrosophy set which involved indeterminacy. Jun et al. [5] introduced neutrosophic cubic set. Senapati et al. [22] studied the cubic subalgebras and cubic closed ideals in detail on B-algebra.

The purpose of this paper is to introduce the idea of neutrosophic cubic MN-subalgebra. We investigate many results to study the neutrosophic cubic MN-subalgebra in detail way by using different concepts like p-intersection, R-intersection and many others.

2. Preliminaries

In this section, some basic definitions are presented that are necessary for this paper.

Definition 2.1 [19] A nonempty set X with a constant 0 and a binary operation $*$ is called BF-algebra, when it fulfills these conditions for all $t_1, t_2 \in X$.

1. $t_1 * t_1 = 0$
2. $t_1 * 0 = t_1$
3. $0 * (t_1 * t_2) = t_2 * t_1$ for all $t_1, t_2 \in X$.

A BF-algebra is denoted by $(X, *, 0)$.

Definition 2.2 [9] A nonempty subset S of BF-algebra X is called a subalgebra of X , if $t_1 * t_2 \in S \forall t_1, t_2 \in S$.

Definition 2.4 [9] A mapping $f : X \rightarrow Y$ of BF-algebra is called homomorphism if $f(t_1 * t_2) = f(t_1) * f(t_2) \forall t_1, t_2 \in X$.

Definition 2.4 [21] Let X be the set of elements which are denoted generally by t_1 . Then a fuzzy set C in X is defined as $C = \{ \langle t_1, v_C(t_1) \rangle \mid t_1 \in X \}$, where $v_C(t_1)$ is called the membership value of t_1 in C and $v_C(t_1) \in [0, 1]$. For a family $C_i = \{ \langle t_1, v_{C_i}(t_1) \rangle \mid t_1 \in X \}$ of fuzzy sets in X , where $i \in U$ and U is index set, they defined the join (\vee) meet (\wedge) operations as:

$$\vee_{i \in U} C_i = (\vee_{i \in U} v_{C_i})(t_1) = \sup\{v_{C_i} \mid i \in U\}$$

and

$$\wedge_{i \in U} C_i = (\wedge_{i \in U} v_{C_i})(t_1) = \inf\{v_{C_i} \mid i \in U\}$$

respectively, $\forall t_1 \in X$.

The finding of supremum and infimum between two intervals is not simple. Biswas [2] explained a procedure to find max/sup and min/inf between two intervals or a set of intervals.

Definition 2.5 [2] Let two elements $P_1, P_2 \in P[0, 1]$. If $P_1 = [(t_1)_1^-, (t_1)_1^+]$ and $P_2 = [(t_1)_2^-, (t_1)_2^+]$, then $\text{rmax}(P_1, P_2) = [\max((t_1)_1^-, (t_1)_2^-), \max((t_1)_1^+, (t_1)_2^+)]$ which is denoted by $P_1 \vee^r P_2$ and $\text{rmin}(P_1, P_2) = [\min((t_1)_1^-, (t_1)_2^-), \min((t_1)_1^+, (t_1)_2^+)]$ which is denoted by $P_1 \wedge^r P_2$. Thus, if $P_i = [((t_1)_i)_i^-, ((t_1)_i)_i^+] \in P[0, 1]$ for $i = 1, 2, 3, \dots$, then we define $\text{rsup}_i(P_i) = [\sup_i(((t_1)_i)_i^-), \sup_i(((t_1)_i)_i^+)]$, i.e., $\vee_i^r P_i = [\vee_i ((t_1)_i)_i^-, \vee_i ((t_1)_i)_i^+]$. In the same way we define $\text{rinfi}_i(P_i) = [\inf_i(((t_1)_i)_i^-), \inf_i(((t_1)_i)_i^+)]$, i.e., $\wedge_i^r P_i = [\wedge_i ((t_1)_i)_i^-, \wedge_i ((t_1)_i)_i^+]$. Now we call $P_1 \geq P_2 \Leftarrow (t_1)_1^- \geq (t_1)_2^-$ and $(t_1)_1^+ \geq (t_1)_2^+$. Similarly the relations $P_1 \leq P_2$ and $P_1 = P_2$ are defined.

Definition 2.6 [1] A fuzzy set $C = \{ \langle t_1, \mu_C(t_1) \rangle \mid t_1 \in X \}$ is called a fuzzy subalgebra of X if $v_C(t_1 * t_2) \geq \min\{v_C(t_1), v_C(t_2)\} \forall t_1, t_2 \in X$.

Jun et al. [5], defined and investigated neutrosophic cubic set.

Definition 2.7 [5] Let X be a nonempty set. A neutrosophic cubic set in X is pair $\mathcal{C} = (\mathfrak{K}, S)$ where $\mathfrak{K} = \{ \langle t_1; \mathfrak{K}_E(t_1), \mathfrak{K}_I(t_1), \mathfrak{K}_N(t_1) \rangle \mid t_1 \in X \}$ is an interval neutrosophic set in X and $S = \{ \langle t_1; S_E(t_1), S_I(t_1), S_N(t_1) \rangle \mid t_1 \in X \}$ is a neutrosophic set in X .

Definition 2.8 [5] For any $\mathcal{C}_i = (\mathfrak{K}_i, S_i)$, where $\mathfrak{K}_i = \{ \langle t_1; \mathfrak{K}_{iE}(t_1), \mathfrak{K}_{iI}(t_1), \mathfrak{K}_{iN}(t_1) \rangle \mid t_1 \in X \}$, $S_i = \{ \langle t_1; S_{iE}(t_1), S_{iI}(t_1), S_{iN}(t_1) \rangle \mid t_1 \in X \}$ for $i \in u$, P-union, P-intersection, R-union and R-intersection are defined respectively by

P-union $\cup_P \mathcal{C}_i = (\cup_{i \in u} \mathfrak{K}_i, \vee_{i \in u} S_i)$, **P-intersection** $\cap_P \mathcal{C}_i = (\cap_{i \in u} \mathfrak{K}_i, \wedge_{i \in u} S_i)$,

R-union $\cup_R \mathcal{C}_i = (\cup_{i \in u} \mathfrak{K}_i, \wedge_{i \in u} S_i)$, **R-intersection** $\cap_R \mathcal{C}_i = (\cap_{i \in u} \mathfrak{K}_i, \vee_{i \in u} S_i)$, where

$$\cup_{i \in U} \mathfrak{N}_i = \{\langle t_1; (\cup_{i \in U} \mathfrak{N}_{iE})(t_1), (\cup_{i \in U} \mathfrak{N}_{iI})(t_1), (\cup_{i \in U} \mathfrak{N}_{iN})(t_1) \rangle | t_1 \in X\},$$

$$\vee_{i \in U} S_i = \{\langle t_1; (\vee_{i \in U} S_{iE})(t_1), (\vee_{i \in U} S_{iI})(t_1), (\vee_{i \in U} S_{iN})(t_1) \rangle | t_1 \in X\},$$

$$\cap_{i \in U} \mathfrak{N}_i = \{\langle t_1; (\cap_{i \in U} \mathfrak{N}_{iE})(t_1), (\cap_{i \in U} \mathfrak{N}_{iI})(t_1), (\cap_{i \in U} \mathfrak{N}_{iN})(t_1) \rangle | t_1 \in X\},$$

$$\wedge_{i \in U} S_i = \{\langle t_1; (\wedge_{i \in U} S_{iE})(t_1), (\wedge_{i \in U} S_{iI})(t_1), (\wedge_{i \in U} S_{iN})(t_1) \rangle | t_1 \in X\}.$$

Definition 2.9 [22] Let $C = \{\langle t_1, \mathfrak{N}(t_1), S(t_1) \rangle\}$ be a cubic set, where $\mathfrak{N}(t_1)$ is an interval-valued fuzzy set in X , $S(t_1)$ is a fuzzy set in X . Then C is cubic subalgebra with following axioms:

$$C1: \mathfrak{N}(t_1 * t_2) \geq \min\{\mathfrak{N}(t_1), \mathfrak{N}(t_2)\},$$

$$C2: S(t_1 * t_2) \leq \max\{S(t_1), S(t_2)\} \forall t_1, t_2 \in X.$$

3. NEUTROSOPHIC CUBIC MN-SUBALGEBRAS

Definition 3.1 Let $\mathfrak{R} = (\mathfrak{N}, S)$ be a cubic set, where X is subalgebra. Then \mathfrak{R} is NCMNSU under binary operation $*$ if it satisfies the following conditions:

$$(N1) \quad \begin{aligned} \mathfrak{N}_E(t_1 * t_2) &\geq \min\{\mathfrak{N}_E(t_1), \mathfrak{N}_E(t_2)\}, \\ \mathfrak{N}_I(t_1 * t_2) &\leq \max\{\mathfrak{N}_I(t_1), \mathfrak{N}_I(t_2)\}, \\ \mathfrak{N}_N(t_1 * t_2) &\leq \max\{\mathfrak{N}_N(t_1), \mathfrak{N}_N(t_2)\}. \end{aligned}$$

$$(N2) \quad \begin{aligned} S_E(t_1 * t_2) &\leq \max\{S_E(t_1), S_E(t_2)\}, \\ S_I(t_1 * t_2) &\leq \min\{S_I(t_1), S_I(t_2)\}, \\ S_N(t_1 * t_2) &\geq \min\{S_N(t_1), S_N(t_2)\}. \end{aligned}$$

Where E means existenceship/membership value, I means indeterminacy existenceship/membership value and N means non existenceship/membership value.

Example 3.1 Let $X = \{0, t_1, t_2, t_3, t_4, t_5\}$ be a BF-algebra with the following Cayley table.

$*$	0	t_1	t_2	t_3	t_4	t_5
0	0	t_5	t_4	t_3	t_2	t_1
t_1	t_1	0	t_5	t_4	t_3	t_2
t_2	t_2	t_1	0	t_5	t_4	t_3
t_3	t_3	t_2	t_1	0	t_5	t_4
t_4	t_4	t_3	t_2	t_1	0	t_5
t_5	t_5	t_4	t_3	t_2	t_1	0

A neutrosophic cubic set $\mathfrak{R} = (\mathfrak{N}_{\overline{E}}, S_{\overline{E}})$ of X is defined by

	0	t_1	t_2	t_3	t_4	t_5
\aleph_E	[0.2,0.4]	[0.1,0.4]	[0.2,0.4]	[0.1,0.4]	[0.2,0.4]	[0.1,0.4]
\aleph_I	[0.7,0.9]	[0.6,0.8]	[0.7,0.9]	[0.6,0.8]	[0.7,0.9]	[0.6,0.8]
\aleph_N	[0.3,0.2]	[0.2,0.1]	[0.3,0.2]	[0.2,0.1]	[0.3,0.2]	[0.2,0.1]

	0	t_1	t_2	t_3	t_4	t_5
S_E	0.1	0.3	0.1	0.3	0.1	0.3
S_I	0.3	0.5	0.3	0.5	0.3	0.5
S_N	0.5	0.6	0.5	0.6	0.5	0.6

All the conditions of Definition 3.1 are satisfied by the set \mathfrak{R} . Thus $\mathfrak{R} = (\aleph_{\Xi}, S_{\Xi})$ is a NCMNSU of X .

Proposition 3.1 Let $\mathfrak{R} = \{ \langle t_1, \aleph_{\Xi}(t_1), S_{\Xi}(t_1) \rangle \}$ is a NCMNSU of X , then $\forall t_1 \in X$, $\aleph_E(t_1) \geq \aleph_E(0)$, $\aleph_I(t_1) \leq \aleph_I(0)$, $\aleph_N(t_1) \leq \aleph_N(0)$ and $S_E(t_1) \leq S_E(0)$, $S_I(t_1) \geq S_I(0)$, $S_N(t_1) \geq S_N(0)$. Thus, $\aleph_{E,I,N}(0)$ and $S_{E,I,N}(0)$ are the upper bound and lower bound of $\aleph_{E,I,N}(t_1)$ and $S_{E,I,N}(t_1)$ respectively.

Proof. $\forall t_1 \in X$, we have $\aleph_E(0) = \aleph_E(t_1 * t_1) \geq \min\{\aleph_E(t_1), \aleph_E(t_1)\} = \aleph_E(t_1) \Rightarrow \aleph_E(0) \geq \aleph_E(t_1)$, $\aleph_I(0) = \aleph_I(t_1 * t_1) \leq \max\{\aleph_I(t_1), \aleph_I(t_1)\} = \aleph_I(t_1) \Rightarrow \aleph_I(0) \leq \aleph_I(t_1)$, $\aleph_N(0) = \aleph_N(t_1 * t_1) \leq \max\{\aleph_N(t_1), \aleph_N(t_1)\} = \aleph_N(t_1) \Rightarrow \aleph_N(0) \leq \aleph_N(t_1)$ and $S_E(0) = S_E(t_1 * t_1) \leq \max\{S_E(t_1), S_E(t_1)\} = S_E(t_1) \Rightarrow S_E(0) \leq S_E(t_1)$, $S_I(0) = S_I(t_1 * t_1) \geq \min\{S_I(t_1), S_I(t_1)\} = S_I(t_1) \Rightarrow S_I(0) \geq S_I(t_1)$, $S_N(0) = S_N(t_1 * t_1) \geq \min\{S_N(t_1), S_N(t_1)\} = S_N(t_1) \Rightarrow S_N(0) \geq S_N(t_1)$.

Theorem 3.1 Let $\mathfrak{R} = \{ \langle t_1, \aleph_{\Xi}(t_1), S_{\Xi}(t_1) \rangle \}$ be a NCMNSU of X . If there exists a sequence $\{(t_1)_n\}$ of X such that $\lim_{n \rightarrow \infty} \aleph_{\Xi}((t_1)_n) = [1,1]$ and $\lim_{n \rightarrow \infty} S_{\Xi}((t_1)_n) = 0$. Then $\aleph_{\Xi}(0) = [1,1]$ and $S_{\Xi}(0) = 0$.

Proof. Using above proposition, $\aleph_E(0) \geq \aleph_E(t_1) \forall t_1 \in X$, $\therefore \aleph_E(0) \geq \aleph_E((t_1)_n)$ for $n \in \mathbb{Z}^+$. Consider, $[1,1] \geq \aleph_E(0) \geq \lim_{n \rightarrow \infty} \aleph_E((t_1)_n) = [1,1]$. So, $\aleph_E(0) = [1,1]$, $\aleph_I(0) \leq \aleph_I(t_1) \forall t_1 \in X$, $\therefore \aleph_I(0) \leq \aleph_I((t_1)_n)$ for $n \in \mathbb{Z}^+$. Consider, $[1,1] \leq \aleph_I(0) \leq \lim_{n \rightarrow \infty} \aleph_I((t_1)_n) = [1,1]$. So, $\aleph_I(0) = [1,1]$, $\aleph_N(0) \leq \aleph_N(t_1) \forall t_1 \in X$, $\therefore \aleph_N(0) \leq \aleph_N((t_1)_n)$ for $n \in \mathbb{Z}^+$. Consider, $[1,1] \leq \aleph_N(0) \leq \lim_{n \rightarrow \infty} \aleph_N((t_1)_n) = [1,1]$. So, $\aleph_N(0) = [1,1]$. Hence, $\aleph_{\Xi}(0) = [1,1]$. Again, using proposition, $S_E(0) \leq S_E(t_1) \forall t_1 \in X$, $\therefore S_E(0) \leq S_E((t_1)_n)$ for $n \in \mathbb{Z}^+$. Consider, $0 \leq S_E(0) \leq \lim_{n \rightarrow \infty} S_E((t_1)_n) = 0$. So, $S_E(0) = 0$, using proposition, $S_I(0) \geq S_I(t_1) \forall t_1 \in X$, $\therefore S_I(0) \geq S_I((t_1)_n)$ for $n \in \mathbb{Z}^+$. Consider, $0 \geq S_I(0) \geq \lim_{n \rightarrow \infty} S_I((t_1)_n) = 0$. So, $S_I(0) = 0$, using proposition, $S_N(0) \geq S_N(t_1) \forall t_1 \in X$, $\therefore S_N(0) \geq S_N((t_1)_n)$ for $n \in \mathbb{Z}^+$. Consider, $0 \geq S_N(0) \geq \lim_{n \rightarrow \infty} S_N((t_1)_n) = 0$. So, $S_N(0) = 0$. Hence, $S_{\Xi}(0) = 0$.

Theorem 3.2 The R-intersection of any set of neutrosophic cubic MN-sun algebra of X is NCMNSU of X .

Proof. Let $\mathfrak{R}_i = \{ \langle t_1, (\aleph_i)_{\Xi}, (S_i)_{\Xi} \rangle | t_1 \in X \}$ where $i \in k$, is family of sets of NCMNSU of X and $t_1, t_2 \in X$. Then $(\cap (\aleph_i)_E)(t_1 * t_2) = \min\{(\aleph_i)_E(t_1 * t_2)\} \geq \min\{\min\{(\aleph_i)_E(t_1), (\aleph_i)_E(t_2)\}\} = \min\{\min\{(\aleph_i)_E(t_1), (\aleph_i)_E(t_2)\}\} = \min\{(\cap (\aleph_i)_E)(t_1), (\cap (\aleph_i)_E)(t_2)\} \Rightarrow (\cap (\aleph_i)_E)(t_1 * t_2) \geq \min\{(\cap (\aleph_i)_E)(t_1), (\cap (\aleph_i)_E)(t_2)\}$, $(\cap (\aleph_i)_I)(t_1 * t_2) = \min\{(\aleph_i)_I(t_1 * t_2)\} \leq \min\{\max\{(\aleph_i)_I(t_1), (\aleph_i)_I(t_2)\}\} = \max\{\min\{(\aleph_i)_I(t_1), (\aleph_i)_I(t_2)\}\} = \max\{(\cap (\aleph_i)_I)(t_1), (\cap (\aleph_i)_I)(t_2)\} \Rightarrow (\cap (\aleph_i)_I)(t_1 * t_2) \leq \max\{(\cap (\aleph_i)_I)(t_1), (\cap (\aleph_i)_I)(t_2)\}$, $(\cap (\aleph_i)_N)(t_1 * t_2) = \min\{(\aleph_i)_N(t_1 * t_2)\} \leq \min\{\max\{(\aleph_i)_N(t_1), (\aleph_i)_N(t_2)\}\} = \max\{\min\{(\aleph_i)_N(t_1), (\aleph_i)_N(t_2)\}\} = \max\{(\cap (\aleph_i)_N)(t_1), (\cap (\aleph_i)_N)(t_2)\} \Rightarrow (\cap (\aleph_i)_N)(t_1 * t_2) \leq \max\{(\cap (\aleph_i)_N)(t_1), (\cap (\aleph_i)_N)(t_2)\}$.

$\leq \text{rinf}\{\text{rmax}\{(\mathfrak{N}_i)_N(t_1), (\mathfrak{N}_i)_N(t_2)\}\} = \text{rmax}\{\text{rinf}(\mathfrak{N}_i)_N(t_1), \text{rinf}(\mathfrak{N}_i)_N(t_2)\} = \text{rmax}\{(\cap (\mathfrak{N}_i)_N)(t_1), (\cap (\mathfrak{N}_i)_N)(t_2)\}$
 $\Rightarrow (\cap (\mathfrak{N}_i)_N)(t_1 * t_2) \leq \text{rmax}\{(\cap (\mathfrak{N}_i)_N)(t_1), (\cap (\mathfrak{N}_i)_N)(t_2)\}$, and $(\vee (S_i)_E)(t_1 * t_2) = \sup (S_i)_E(t_1 * t_2) \leq \sup \{$
 $\text{max}\{(S_i)_E(t_1), (S_i)_E(t_2)\}\} = \max \{\sup (S_i)_E(t_1), \sup (S_i)_E(t_2)\} = \max \{(\vee (S_i)_E)(t_1), (\vee (S_i)_E)(t_2)\} \Rightarrow (\vee (S_i)_E)$
 $(t_1 * t_2) \leq \max \{(\vee (S_i)_E)(t_1), (\vee (S_i)_E)(t_2)\}$, $(\vee (S_i)_I)(t_1 * t_2) = \sup (S_i)_I(t_1 * t_2) \geq \sup \{\min \{(S_i)_I(t_1), (S_i)_I(t_2)\}\}$
 $\} = \min \{\sup (S_i)_I(t_1), \sup (S_i)_I(t_2)\} = \min \{(\vee (S_i)_I)(t_1), (\vee (S_i)_I)(t_2)\} \Rightarrow (\vee (S_i)_I)(t_1 * t_2) \geq \min \{(\vee (S_i)_I)(t_1), (\vee (S_i)_I)(t_2)\}$,
 $(\vee (S_i)_N)(t_1 * t_2) = \sup (S_i)_N(t_1 * t_2) \leq \sup \{\min \{(S_i)_N(t_1), (S_i)_N(t_2)\}\} = \min \{\sup (S_i)_N(t_1), \sup (S_i)_N(t_2)\}$
 $= \min \{(\vee (S_i)_N)(t_1), (\vee (S_i)_N)(t_2)\} \Rightarrow (\vee (S_i)_N)(t_1 * t_2) \leq \min \{(\vee (S_i)_N)(t_1), (\vee (S_i)_N)(t_2)\}$,
 which show that R-intersection of \mathfrak{R}_i is NCMNSU of X.

Theorem 3.3 Let $\mathfrak{R}_i = \{(t_1, (\mathfrak{N}_i)_E, (S_i)_E) | t_1 \in X\}$ be a collection of sets of NCMNSU of X, where $i \in k$. If $\inf \{\max \{(S_i)_E(t_1), (S_i)_E(t_2)\}\} = \max \{\inf (S_i)_E(t_1), \inf (S_i)_E(t_2)\}$, $\inf \{\min \{(S_i)_I(t_1), (S_i)_I(t_2)\}\} = \min \{\inf (S_i)_I(t_1), \inf (S_i)_I(t_2)\}$, $\inf \{\min \{(S_i)_N(t_1), (S_i)_N(t_2)\}\} = \min \{\inf (S_i)_N(t_1), \inf (S_i)_N(t_2)\} \forall t_1 \in X$, then the P-intersection of \mathfrak{R}_i is also a NCMNSU of X.

Proof. Suppose that $\mathfrak{R}_i = \{(t_1, (\mathfrak{N}_i)_E, (S_i)_E) | t_1 \in X\}$ where $i \in k$, be a collection of sets of NCMNSU of X such that $\inf \{\max \{(S_i)_E(t_1), (S_i)_E(t_2)\}\} = \max \{\inf (S_i)_E(t_1), \inf (S_i)_E(t_2)\}$, $\inf \{\min \{(S_i)_I(t_1), (S_i)_I(t_2)\}\} = \min \{\inf (S_i)_I(t_1), \inf (S_i)_I(t_2)\}$, $\inf \{\min \{(S_i)_N(t_1), (S_i)_N(t_2)\}\} = \min \{\inf (S_i)_N(t_1), \inf (S_i)_N(t_2)\} \forall t_1 \in X$. Now for $t_1, t_2 \in X$. Then $(\cap (\mathfrak{N}_i)_E)(t_1 * t_2) = \text{rinf}(\mathfrak{N}_i)_E(t_1 * t_2) \geq \text{rinf}\{\text{rmin}\{(\mathfrak{N}_i)_E(t_1), (\mathfrak{N}_i)_E(t_2)\}\} = \text{rmin}\{\text{rinf}(\mathfrak{N}_i)_E(t_1), \text{rinf}(\mathfrak{N}_i)_E(t_2)\} = \text{rmin}\{(\cap (\mathfrak{N}_i)_E)(t_1), (\cap (\mathfrak{N}_i)_E)(t_2)\} \Rightarrow (\cap (\mathfrak{N}_i)_E)(t_1 * t_2) \geq \text{rmin}\{(\cap (\mathfrak{N}_i)_E)(t_1), (\cap (\mathfrak{N}_i)_E)(t_2)\}$, $(\cap (\mathfrak{N}_i)_I)(t_1 * t_2) = \text{rinf}(\mathfrak{N}_i)_I(t_1 * t_2) \leq \text{rinf}\{\text{rmax}\{(\mathfrak{N}_i)_I(t_1), (\mathfrak{N}_i)_I(t_2)\}\} = \text{rmax}\{\text{rinf}(\mathfrak{N}_i)_I(t_1), \text{rinf}(\mathfrak{N}_i)_I(t_2)\} = \text{rmax}\{(\cap (\mathfrak{N}_i)_I)(t_1), (\cap (\mathfrak{N}_i)_I)(t_2)\} \Rightarrow (\cap (\mathfrak{N}_i)_I)(t_1 * t_2) \leq \text{rmax}\{(\cap (\mathfrak{N}_i)_I)(t_1), (\cap (\mathfrak{N}_i)_I)(t_2)\}$, $(\cap (\mathfrak{N}_i)_N)(t_1 * t_2) = \text{rinf}(\mathfrak{N}_i)_N(t_1 * t_2) \leq \text{rinf}\{\text{rmax}\{(\mathfrak{N}_i)_N(t_1), (\mathfrak{N}_i)_N(t_2)\}\} = \text{rmax}\{\text{rinf}(\mathfrak{N}_i)_N(t_1), \text{rinf}(\mathfrak{N}_i)_N(t_2)\} = \text{rmax}\{(\cap (\mathfrak{N}_i)_N)(t_1), (\cap (\mathfrak{N}_i)_N)(t_2)\} \Rightarrow (\cap (\mathfrak{N}_i)_N)(t_1 * t_2) \leq \text{rmax}\{(\cap (\mathfrak{N}_i)_N)(t_1), (\cap (\mathfrak{N}_i)_N)(t_2)\}$, and $(\wedge (S_i)_E)(t_1 * t_2) = \inf (S_i)_E(t_1 * t_2) \leq \inf \{\max \{(S_i)_E(t_1), (S_i)_E(t_2)\}\} = \max \{\inf (S_i)_E(t_1), \inf (S_i)_E(t_2)\} = \max \{(\wedge (S_i)_E)(t_1), (\wedge (S_i)_E)(t_2)\} \Rightarrow (\wedge (S_i)_E)(t_1 * t_2) \leq \max \{(\wedge (S_i)_E)(t_1), (\wedge (S_i)_E)(t_2)\}$, $(\wedge (S_i)_I)(t_1 * t_2) = \inf (S_i)_I(t_1 * t_2) \geq \inf \{\min \{(S_i)_I(t_1), (S_i)_I(t_2)\}\} = \min \{\inf (S_i)_I(t_1), \inf (S_i)_I(t_2)\} = \min \{(\wedge (S_i)_I)(t_1), (\wedge (S_i)_I)(t_2)\} \Rightarrow (\wedge (S_i)_I)(t_1 * t_2) \geq \min \{(\wedge (S_i)_I)(t_1), (\wedge (S_i)_I)(t_2)\}$, $(\wedge (S_i)_N)(t_1 * t_2) = \inf (S_i)_N(t_1 * t_2) \geq \inf \{\min \{(S_i)_N(t_1), (S_i)_N(t_2)\}\} = \min \{\inf (S_i)_N(t_1), \inf (S_i)_N(t_2)\} = \min \{(\wedge (S_i)_N)(t_1), (\wedge (S_i)_N)(t_2)\} \Rightarrow (\wedge (S_i)_N)(t_1 * t_2) \geq \min \{(\wedge (S_i)_N)(t_1), (\wedge (S_i)_N)(t_2)\}$. Which show that P-intersection of \mathfrak{R}_i is NCMNSU of X.

Theorem 3.4 Let $\mathfrak{R}_i = \{(t_1, (\mathfrak{N}_i)_E, (S_i)_E) | t_1 \in X\}$ where $i \in k$, be a collection of sets of NCMNSU of X. If $\text{rsup}\{\text{rmin}\{(\mathfrak{N}_i)_E(t_1), (\mathfrak{N}_i)_E(t_2)\}\} = \text{rmin}\{\text{rsup}(\mathfrak{N}_i)_E(t_1), \text{rsup}(\mathfrak{N}_i)_E(t_2)\}$, $\text{rsup}\{\text{rmax}\{(\mathfrak{N}_i)_I(t_1), (\mathfrak{N}_i)_I(t_2)\}\} = \text{rmax}\{\text{rsup}(\mathfrak{N}_i)_I(t_1), \text{rsup}(\mathfrak{N}_i)_I(t_2)\}$, $\text{rsup}\{\text{rmax}\{(\mathfrak{N}_i)_N(t_1), (\mathfrak{N}_i)_N(t_2)\}\} = \text{rmax}\{\text{rsup}(\mathfrak{N}_i)_N(t_1), \text{rsup}(\mathfrak{N}_i)_N(t_2)\}$, and $\sup \{\max \{(S_i)_E(t_1), (S_i)_E(t_2)\}\} = \max \{\sup (S_i)_E(t_1), \sup (S_i)_E(t_2)\}$, $\sup \{\min \{(S_i)_I(t_1), (S_i)_I(t_2)\}\} = \min \{\sup (S_i)_I(t_1), \sup (S_i)_I(t_2)\}$, $\sup \{\min \{(S_i)_N(t_1), (S_i)_N(t_2)\}\} = \min \{\sup (S_i)_N(t_1), \sup (S_i)_N(t_2)\} \forall t_1, t_2 \in X$. Then P-union of \mathfrak{R}_i is NCMNSU of X.

Proof. Let $\mathfrak{R}_i = \{(t_1, (\mathfrak{N}_i)_E, (S_i)_E) | t_1 \in X\}$ where $i \in k$, be a collection of sets of NCMNSU of X. $\forall t_1, t_2 \in X$, we have some conditions mentioned in theorem. Then for $t_1, t_2 \in X$. $(\cup (\mathfrak{N}_i)_E)(t_1 * t_2) = \text{rsup}(\mathfrak{N}_i)_E(t_1 * t_2) \geq \text{rsup}\{\text{rmin}\{(\mathfrak{N}_i)_E(t_1), (\mathfrak{N}_i)_E(t_2)\}\} = \text{rmin}\{\text{rsup}(\mathfrak{N}_i)_E(t_1), \text{rsup}(\mathfrak{N}_i)_E(t_2)\} = \text{rmin}\{(\cup (\mathfrak{N}_i)_E)(t_1), (\cup (\mathfrak{N}_i)_E)(t_2)\} \Rightarrow (\cup (\mathfrak{N}_i)_E)(t_1 * t_2) \geq \text{rmin}\{(\cup (\mathfrak{N}_i)_E)(t_1), (\cup (\mathfrak{N}_i)_E)(t_2)\}$, $(\cup (\mathfrak{N}_i)_I)(t_1 * t_2) = \text{rsup}(\mathfrak{N}_i)_I(t_1 * t_2) \leq \text{rsup}\{\text{rmax}\{(\mathfrak{N}_i)_I(t_1), (\mathfrak{N}_i)_I(t_2)\}\} = \text{rmax}\{\text{rsup}(\mathfrak{N}_i)_I(t_1), \text{rsup}(\mathfrak{N}_i)_I(t_2)\} = \text{rmax}\{(\cup (\mathfrak{N}_i)_I)(t_1), (\cup (\mathfrak{N}_i)_I)(t_2)\} \Rightarrow (\cup (\mathfrak{N}_i)_I)(t_1 * t_2) \leq \text{rmax}\{(\cup (\mathfrak{N}_i)_I)(t_1), (\cup (\mathfrak{N}_i)_I)(t_2)\}$, $(\cup (\mathfrak{N}_i)_N)(t_1 * t_2) = \text{rsup}(\mathfrak{N}_i)_N(t_1 * t_2) \leq \text{rsup}\{\text{rmax}\{(\mathfrak{N}_i)_N(t_1), (\mathfrak{N}_i)_N(t_2)\}\} = \text{rmax}\{\text{rsup}(\mathfrak{N}_i)_N(t_1), \text{rsup}(\mathfrak{N}_i)_N(t_2)\} = \text{rmax}\{(\cup (\mathfrak{N}_i)_N)(t_1), (\cup (\mathfrak{N}_i)_N)(t_2)\} \Rightarrow (\cup (\mathfrak{N}_i)_N)(t_1 * t_2) \leq \text{rmax}\{(\cup (\mathfrak{N}_i)_N)(t_1), (\cup (\mathfrak{N}_i)_N)(t_2)\}$, and $(\vee (S_i)_E)(t_1 * t_2) = \sup (S_i)_E(t_1 * t_2) \leq \sup \{\max \{(S_i)_E(t_1), (S_i)_E(t_2)\}\} = \max \{\sup (S_i)_E(t_1), \sup (S_i)_E(t_2)\} = \max \{(\vee (S_i)_E)(t_1), (\vee (S_i)_E)(t_2)\} \Rightarrow (\vee (S_i)_E)(t_1 * t_2) \leq \max \{(\vee (S_i)_E)(t_1), (\vee (S_i)_E)(t_2)\}$, $(\vee (S_i)_I)(t_1 * t_2) = \sup (S_i)_I(t_1 * t_2) \geq \sup \{\min \{(S_i)_I(t_1), (S_i)_I(t_2)\}\} = \min \{\sup (S_i)_I(t_1), \sup (S_i)_I(t_2)\} = \min \{(\vee (S_i)_I)(t_1), (\vee (S_i)_I)(t_2)\} \Rightarrow (\vee (S_i)_I)(t_1 * t_2) \geq \min \{(\vee (S_i)_I)(t_1), (\vee (S_i)_I)(t_2)\}$, $(\vee (S_i)_N)(t_1 * t_2) = \sup (S_i)_N(t_1 * t_2) \geq \sup \{\min \{(S_i)_N(t_1), (S_i)_N(t_2)\}\} = \min \{\sup (S_i)_N(t_1), \sup (S_i)_N(t_2)\} = \min \{(\vee (S_i)_N)(t_1), (\vee (S_i)_N)(t_2)\} \Rightarrow (\vee (S_i)_N)(t_1 * t_2) \geq \min \{(\vee (S_i)_N)(t_1), (\vee (S_i)_N)(t_2)\}$.

$t_1 * t_2 = \sup (S_i)_N(t_1 * t_2) \geq \sup \{\min \{(S_i)_N(t_1), (S_i)_N(t_2)\}\} = \min \{\sup (S_i)_N(t_1), \sup (S_i)_N(t_2)\} = \min \{(\vee (S_i)_N)(t_1), (\vee (S_i)_N)(t_2)\} \Rightarrow (\vee (S_i)_N)(t_1 * t_2) \geq \min\{(\vee (S_i)_N)(t_1), (\vee (S_i)_N)(t_2)\}$, which show that P-union of \mathfrak{R}_i is NCMNSU of X .

Theorem 3.5 If neutrosophic cubic set $\mathfrak{R} = (\mathfrak{N}_E, S_E)$ of X is subalgebra, then $\forall t_1 \in X, \mathfrak{N}_E(0 * t_1) \geq \mathfrak{N}_E(t_1), \mathfrak{N}_I(0 * t_1) \leq \mathfrak{N}_I(t_1), \mathfrak{N}_N(0 * t_1) \leq \mathfrak{N}_N(t_1)$ and $S_E(0 * t_1) \leq S_E(t_1), S_I(0 * t_1) \geq S_I(t_1), S_N(0 * t_1) \geq S_N(t_1)$.

Proof. $\forall t_1 \in X, \mathfrak{N}_E(0 * t_1) \geq \text{rmin}\{\mathfrak{N}_E(0), \mathfrak{N}_E(t_1)\} = \text{rmin}\{\mathfrak{N}_E(t_1 * t_1), \mathfrak{N}_E(t_1)\} \geq \text{rmin}\{\text{rmin}\{\mathfrak{N}_E(t_1), \mathfrak{N}_E(t_1)\}, \mathfrak{N}_E(t_1)\} = \mathfrak{N}_E(t_1), \mathfrak{N}_I(0 * t_1) \leq \text{rmax}\{\mathfrak{N}_I(0), \mathfrak{N}_I(t_1)\} = \text{rmax}\{\mathfrak{N}_I(t_1 * t_1), \mathfrak{N}_I(t_1)\} \leq \text{rmax}\{\text{rmax}\{\mathfrak{N}_I(t_1), \mathfrak{N}_I(t_1)\}, \mathfrak{N}_I(t_1)\} = \mathfrak{N}_I(t_1), \mathfrak{N}_N(0 * t_1) \leq \text{rmax}\{\mathfrak{N}_N(0), \mathfrak{N}_N(t_1)\} = \text{rmax}\{\mathfrak{N}_N(t_1 * t_1), \mathfrak{N}_N(t_1)\} \leq \text{rmax}\{\text{rmax}\{\mathfrak{N}_N(t_1), \mathfrak{N}_N(t_1)\}, \mathfrak{N}_N(t_1)\} = \mathfrak{N}_N(t_1)$ and now $\forall t_1 \in X, S_E(0 * t_1) \leq \text{max}\{S_E(0), S_E(t_1)\} = \text{max}\{S_E(t_1 * t_1), S_E(t_1)\} \leq \text{max}\{\text{max}\{S_E(t_1), S_E(t_1)\}, S_E(t_1)\} = S_E(t_1), S_I(0 * t_1) \geq \text{min}\{S_I(0), S_I(t_1)\} = \text{min}\{S_I(t_1 * t_1), S_I(t_1)\} \geq \text{min}\{\text{min}\{S_I(t_1), S_I(t_1)\}, S_I(t_1)\} = S_I(t_1), S_N(0 * t_1) \geq \text{min}\{S_N(0), S_N(t_1)\} = \text{min}\{S_N(t_1 * t_1), S_N(t_1)\} \geq \text{min}\{\text{min}\{S_N(t_1), S_N(t_1)\}, S_N(t_1)\} = S_N(t_1)$

Theorem 3.6 If neutrosophic cubic set $\mathfrak{R} = (\mathfrak{N}_E, S_E)$ of X is subalgebra then $\mathfrak{R}(t_1 * t_2) = \mathfrak{R}(t_1 * (0 * (0 * t_2))) \forall t_1, t_2 \in X$.

Proof. Let X be a BF-algebra and $t_1, t_2 \in X$. Then we know by ([13] Proposition 2.5) that $t_2 = 0 * (0 * t_2)$. Hence, $\mathfrak{N}_E(t_1 * t_2) = \mathfrak{N}_E(t_1 * (0 * (0 * t_2)))$ and $S_E(t_1 * t_2) = S_E(t_1 * (0 * (0 * t_2)))$. Therefore, $\mathfrak{R}_E(t_1 * t_2) = \mathfrak{R}_E(t_1 * (0 * (0 * t_2)))$.

Theorem 3.7 If neutrosophic cubic set $\mathfrak{R} = (\mathfrak{N}_E, S_E)$ of X is subalgebra then $\mathfrak{N}_E(0) \geq \text{rmin}\{\mathfrak{N}_E(t_2), \mathfrak{N}_E(t_1)\}, \mathfrak{N}_I(0) \leq \text{rmax}\{\mathfrak{N}_I(t_2), \mathfrak{N}_I(t_1)\}, \mathfrak{N}_N(0) \leq \text{rmax}\{\mathfrak{N}_N(t_2), \mathfrak{N}_N(t_1)\}$, and $S_E(0) \leq \text{max}\{S_E(t_2), S_E(t_1)\}, S_I(0) \geq \text{min}\{S_I(t_2), S_I(t_1)\}, S_N(0) \geq \text{min}\{S_N(t_2), S_N(t_1)\}, \forall t_1, t_2 \in X$.

Proof. Here we use the ([13] Proposition 2.5) and above Proposition, now for $t_1, t_2 \in X \mathfrak{N}_E(0) = \mathfrak{N}_E(t_1 * t_1) \geq \text{rmax}\{\mathfrak{N}_E(t_1), \mathfrak{N}_E(t_1)\} = \text{rmin}\{\mathfrak{N}_E(0 * (0 * t_1)), \mathfrak{N}_E(0 * (0 * t_1))\} = \text{rmin}\{\mathfrak{N}_E(0 * (0 * t_2)), \mathfrak{N}_E(0 * (0 * t_1))\} = \text{rmin}\{\mathfrak{N}_E(t_2), \mathfrak{N}_E(t_1)\}, \mathfrak{N}_I(0) = \mathfrak{N}_I(t_1 * t_1) \leq \text{rmax}\{\mathfrak{N}_I(t_1), \mathfrak{N}_I(t_1)\} = \text{rmax}\{\mathfrak{N}_I(0 * (0 * t_1)), \mathfrak{N}_I(0 * (0 * t_1))\} = \text{rmax}\{\mathfrak{N}_I(0 * (0 * t_2)), \mathfrak{N}_I(0 * (0 * t_1))\} = \text{rmax}\{\mathfrak{N}_I(t_2), \mathfrak{N}_I(t_1)\}, \mathfrak{N}_N(0) = \mathfrak{N}_N(t_1 * t_1) \leq \text{rmax}\{\mathfrak{N}_N(t_1), \mathfrak{N}_N(t_1)\} = \text{rmax}\{\mathfrak{N}_N(0 * (0 * t_1)), \mathfrak{N}_N(0 * (0 * t_1))\} = \text{rmax}\{\mathfrak{N}_N(0 * (0 * t_2)), \mathfrak{N}_N(0 * (0 * t_1))\} = \text{rmax}\{\mathfrak{N}_N(t_2), \mathfrak{N}_N(t_1)\}$. Now, $S_E(0) = S_E(t_1 * t_1) \leq \text{max}\{S_E(t_1), S_E(t_1)\} = \text{max}\{S_E(0 * (0 * t_1)), S_E(0 * (0 * t_1))\} = \text{max}\{S_E(0 * (0 * t_2)), S_E(0 * (0 * t_1))\} = \text{max}\{S_E(t_2), S_E(t_1)\}, S_I(0) = S_I(t_1 * t_1) \geq \text{min}\{S_I(t_1), S_I(t_1)\} = \text{min}\{S_I(0 * (0 * t_1)), S_I(0 * (0 * t_1))\} = \text{min}\{S_I(0 * (0 * t_2)), S_I(0 * (0 * t_1))\} = \text{min}\{S_I(t_2), S_I(t_1)\}, S_N(0) = S_N(t_1 * t_1) \geq \text{min}\{S_N(t_1), S_N(t_1)\} = \text{min}\{S_N(0 * (0 * t_1)), S_N(0 * (0 * t_1))\} = \text{min}\{S_N(0 * (0 * t_2)), S_N(0 * (0 * t_1))\} = \text{min}\{S_N(t_2), S_N(t_1)\}.$

Theorem 3.8 If neutrosophic cubic set $\mathfrak{R} = (\mathfrak{N}_E, S_E)$ of X is NCMNSU, then $\forall t_1, t_2 \in X, \mathfrak{N}_E(t_1 * (0 * t_2)) \geq \text{rmin}\{\mathfrak{N}_E(t_1), \mathfrak{N}_E(t_2)\}$ and $S_E(t_1 * (0 * t_2)) \leq \text{max}\{S_E(t_1), S_E(t_2)\}$.

Proof. Here we use above Proposition for proof. Let $t_1, t_2 \in X$. Then we have $\mathfrak{N}_E(t_1 * (0 * t_2)) \geq \text{rmin}\{\mathfrak{N}_E(t_1), \mathfrak{N}_E(0 * t_2)\} \geq \text{rmin}\{\mathfrak{N}_E(t_1), \mathfrak{N}_E(t_2)\}, \mathfrak{N}_I(t_1 * (0 * t_2)) \leq \text{rmax}\{\mathfrak{N}_I(t_1), \mathfrak{N}_I(0 * t_2)\} \leq \text{rmax}\{\mathfrak{N}_I(t_1), \mathfrak{N}_I(t_2)\}, \mathfrak{N}_N(t_1 * (0 * t_2)) \leq \text{rmax}\{\mathfrak{N}_N(t_1), \mathfrak{N}_N(0 * t_2)\} \leq \text{rmax}\{\mathfrak{N}_N(t_1), \mathfrak{N}_N(t_2)\}$ and $S_E(t_1 * (0 * t_2)) \leq \text{max}\{S_E(t_1),$

$S_E(0 * t_2)\} \leq \text{max}\{S_E(t_1), S_E(t_2)\}, S_I(t_1 * (0 * t_2)) \geq \text{min}\{S_I(t_1), S_I(0 * t_2)\} \geq \text{min}\{S_I(t_1), S_I(t_2)\}, S_N(t_1 * (0 * t_2)) \geq \text{min}\{S_N(t_1), S_N(0 * t_2)\} \geq \text{min}\{S_N(t_1), S_N(t_2)\}.$

Theorem 3.9 If a neutrosophic cubic set $\mathfrak{R} = (\mathfrak{N}_E, S_E)$ of X satisfies the following conditions, then \mathfrak{R} refers to a NCMNSU of X :

1. $\aleph_E(0 * t_1) \geq \aleph_E(t_1)$, $\aleph_I(0 * t_1) \leq \aleph_I(t_1)$, $\aleph_N(0 * t_1) \leq \aleph_N(t_1)$ and $S_E(0 * t_1) \leq S_E(t_1)$, $S_I(0 * t_1) \geq S_I(t_1)$, $S_N(0 * t_1) \geq S_N(t_1) \forall t_1 \in X$.

2. $\aleph_E(t_1 * (0 * t_2)) \geq \text{rmin}\{\aleph_E(t_1), \aleph_E(t_2)\}$, $\aleph_I(t_1 * (0 * t_2)) \leq \text{rmax}\{\aleph_I(t_1), \aleph_I(t_2)\}$, $\aleph_N(t_1 * (0 * t_2)) \leq \text{rmax}\{\aleph_N(t_1), \aleph_N(t_2)\}$ and $S_E(t_1 * (0 * t_2)) \leq \text{max}\{S_E(t_1), S_E(t_2)\}$, $S_I(t_1 * (0 * t_2)) \geq \text{min}\{S_I(t_1), S_I(t_2)\}$, $S_N(t_1 * (0 * t_2)) \geq \text{min}\{S_N(t_1), S_N(t_2)\}$, $\forall t_1, t_2 \in X$.

Proof. Assume that the neutrosophic cubic set $\mathfrak{R} = (\aleph_{\Xi}, S_{\Xi})$ of X satisfies the both axioms above. Then by lemma, we have $\aleph_E(t_1 * t_2) = \aleph_E(t_1 * (0 * (0 * t_2))) \geq \text{rmin}\{\aleph_E(t_1), \aleph_E(0 * t_2)\} \geq \text{rmin}\{\aleph_E(t_1), \aleph_E(t_2)\}$, $\aleph_I(t_1 * t_2) = \aleph_I(t_1 * (0 * (0 * t_2))) \leq \text{rmax}\{\aleph_I(t_1), \aleph_I(0 * t_2)\} \leq \text{rmax}\{\aleph_I(t_1), \aleph_I(t_2)\}$, $\aleph_N(t_1 * t_2) = \aleph_N(t_1 * (0 * (0 * t_2))) \leq \text{rmax}\{\aleph_N(t_1), \aleph_N(0 * t_2)\} \leq \text{rmax}\{\aleph_N(t_1), \aleph_N(t_2)\}$, and $S_E(t_1 * t_2) = S_E(t_1 * (0 * (0 * t_2))) \leq \text{max}\{S_E(t_1), S_E(0 * t_2)\} \leq \text{max}\{S_E(t_1), S_E(t_2)\}$, $S_I(t_1 * t_2) = S_I(t_1 * (0 * (0 * t_2))) \geq \text{min}\{S_I(t_1), S_I(0 * t_2)\} \geq \text{min}\{S_I(t_1), S_I(t_2)\}$, $S_N(t_1 * t_2) = S_N(t_1 * (0 * (0 * t_2))) \geq \text{min}\{S_N(t_1), S_N(0 * t_2)\} \geq \text{min}\{S_N(t_1), S_N(t_2)\} \forall t_1, t_2 \in X$. Hence, \mathfrak{R} is NCMNSU of X .

Theorem 3.10 A neutrosophic cubic set $\mathfrak{R} = (\aleph_{\Xi}, S_{\Xi})$ of X is NCMNSU of $X \Leftarrow \aleph_{\Xi}^-, \aleph_{\Xi}^+$ and S_{Ξ} are fuzzy subalgebra of X .

Proof. Let $\aleph_{\Xi}^-, \aleph_{\Xi}^+$ and S_{Ξ} are fuzzy subalgebra of X and $t_1, t_2 \in X$. Then $\aleph_E^-(t_1 * t_2) \geq \text{min}\{\aleph_E^-(t_1), \aleph_E^-(t_2)\}$, $\aleph_I^-(t_1 * t_2) \leq \text{max}\{\aleph_I^-(t_1), \aleph_I^-(t_2)\}$, $\aleph_N^-(t_1 * t_2) \leq \text{max}\{\aleph_N^-(t_1), \aleph_N^-(t_2)\}$, $\aleph_E^+(t_1 * t_2) \geq \text{min}\{\aleph_E^+(t_1), \aleph_E^+(t_2)\}$, $\aleph_I^+(t_1 * t_2) \leq \text{max}\{\aleph_I^+(t_1), \aleph_I^+(t_2)\}$, $\aleph_N^+(t_1 * t_2) \leq \text{max}\{\aleph_N^+(t_1), \aleph_N^+(t_2)\}$, and $S_E(t_1 * t_2) \leq \text{max}\{S_E(t_1), S_E(t_2)\}$, $S_I(t_1 * t_2) \geq \text{min}\{S_I(t_1), S_I(t_2)\}$, $S_N(t_1 * t_2) \geq \text{min}\{S_N(t_1), S_N(t_2)\}$. Now, $\aleph_E(t_1 * t_2) = [\aleph_E^-(t_1 * t_2), \aleph_E^+(t_1 * t_2)] \geq [\text{min}\{\aleph_E^-(t_1), \aleph_E^-(t_2)\}, \text{min}\{\aleph_E^+(t_1), \aleph_E^+(t_2)\}] \geq \text{rmin}\{[\aleph_E^-(t_1), \aleph_E^+(t_1)], [\aleph_E^-(t_2), \aleph_E^+(t_2)]\} = \text{rmin}\{\aleph_E(t_1), \aleph_E(t_2)\}$, $\aleph_I(t_1 * t_2) = [\aleph_I^-(t_1 * t_2), \aleph_I^+(t_1 * t_2)] \leq [\text{max}\{\aleph_I^-(t_1), \aleph_I^-(t_2)\}, \text{max}\{\aleph_I^+(t_1), \aleph_I^+(t_2)\}] \leq \text{rmax}\{[\aleph_I^-(t_1), \aleph_I^+(t_1)], [\aleph_I^-(t_2), \aleph_I^+(t_2)]\} = \text{rmax}\{\aleph_I(t_1), \aleph_I(t_2)\}$, $\aleph_N(t_1 * t_2) = [\aleph_N^-(t_1 * t_2), \aleph_N^+(t_1 * t_2)] \leq [\text{max}\{\aleph_N^-(t_1), \aleph_N^-(t_2)\}, \text{max}\{\aleph_N^+(t_1), \aleph_N^+(t_2)\}] \leq \text{rmax}\{[\aleph_N^-(t_1), \aleph_N^+(t_1)], [\aleph_N^-(t_2), \aleph_N^+(t_2)]\} = \text{rmax}\{\aleph_N(t_1), \aleph_N(t_2)\}$. Therefore, \mathfrak{R} is NCMNSU of X . Conversely, assume that \mathfrak{R} is a NCMNSU of X . For any $t_1, t_2 \in X$, $\{\aleph_E^-(t_1 * t_2), \aleph_E^+(t_1 * t_2)\} = \aleph_E(t_1 * t_2) \geq \text{rmin}\{\aleph_E(t_1), \aleph_E(t_2)\} = \text{rmin}\{[\aleph_E^-(t_1), \aleph_E^+(t_1)], [\aleph_E^-(t_2), \aleph_E^+(t_2)]\} = [\text{min}\{\aleph_E^-(t_1), \aleph_E^-(t_2)\}, \text{min}\{\aleph_E^+(t_1), \aleph_E^+(t_2)\}]$, $\{\aleph_I^-(t_1 * t_2), \aleph_I^+(t_1 * t_2)\} = \aleph_I(t_1 * t_2) \leq \text{rmax}\{\aleph_I(t_1), \aleph_I(t_2)\} = \text{rmax}\{[\aleph_I^-(t_1), \aleph_I^+(t_1)], [\aleph_I^-(t_2), \aleph_I^+(t_2)]\} = [\text{max}\{\aleph_I^-(t_1), \aleph_I^-(t_2)\}, \text{max}\{\aleph_I^+(t_1), \aleph_I^+(t_2)\}]$, $\{\aleph_N^-(t_1 * t_2), \aleph_N^+(t_1 * t_2)\} = \aleph_N(t_1 * t_2) \leq \text{rmax}\{\aleph_N(t_1), \aleph_N(t_2)\} = \text{rmax}\{[\aleph_N^-(t_1), \aleph_N^+(t_1)], [\aleph_N^-(t_2), \aleph_N^+(t_2)]\} = [\text{max}\{\aleph_N^-(t_1), \aleph_N^-(t_2)\}, \text{max}\{\aleph_N^+(t_1), \aleph_N^+(t_2)\}]$. Thus, $\aleph_E^-(t_1 * t_2) \geq \text{min}\{\aleph_E^-(t_1), \aleph_E^-(t_2)\}$, $\aleph_I^-(t_1 * t_2) \leq \text{max}\{\aleph_I^-(t_1), \aleph_I^-(t_2)\}$, $\aleph_N^-(t_1 * t_2) \leq \text{max}\{\aleph_N^-(t_1), \aleph_N^-(t_2)\}$, $\aleph_E^+(t_1 * t_2) \geq \text{min}\{\aleph_E^+(t_1), \aleph_E^+(t_2)\}$, $\aleph_I^+(t_1 * t_2) \leq \text{max}\{\aleph_I^+(t_1), \aleph_I^+(t_2)\}$, $\aleph_N^+(t_1 * t_2) \leq \text{max}\{\aleph_N^+(t_1), \aleph_N^+(t_2)\}$, and $S_E(t_1 * t_2) \leq \text{max}\{S_E(t_1), S_E(t_2)\}$, $S_I(t_1 * t_2) \geq \text{min}\{S_I(t_1), S_I(t_2)\}$, $S_N(t_1 * t_2) \geq \text{min}\{S_N(t_1), S_N(t_2)\}$. Hence $\aleph_{\Xi}^-, \aleph_{\Xi}^+$ and S_{Ξ} are fuzzy subalgebra of X .

Theorem 3.11 Let $\mathfrak{R} = (\aleph_{\Xi}, S_{\Xi})$ be a NCMNSU of X and $n \in \mathbb{Z}^+$ (the set of positive integer). Then 1. $\aleph_E(7^n t_1 * t_1) \geq \aleph_E(t_1)$ for $n \in \mathbb{O}$. 2. $\aleph_I(7^n t_1 * t_1) \leq \aleph_I(t_1)$ for $n \in \mathbb{O}$. 3. $\aleph_N(7^n t_1 * t_1) \leq \aleph_N(t_1)$ for $n \in \mathbb{O}$. 4. $S_E(7^n t_1 * t_1) \leq \aleph_E(t_1)$ for $n \in \mathbb{O}$. 5. $S_I(7^n t_1 * t_1) \geq \aleph_I(t_1)$ for $n \in \mathbb{O}$. 6. $S_N(7^n t_1 * t_1) \geq \aleph_N(t_1)$ for $n \in \mathbb{O}$. 7. $\aleph_{\Xi}(7^n t_1 * t_1) = \aleph_{\Xi}(t_1)$ for $n \in \mathbb{E}$. 8. $S_{\Xi}(7^n t_1 * t_1) = S_{\Xi}(t_1)$ for $n \in \mathbb{E}$.

Proof. Let $t_1 \in X$ and n is odd. Then $n = 2q - 1$ for some positive integer q . We prove the theorem by induction.

Now $\aleph_E(t_1 * t_1) = \aleph_E(0) \geq \aleph_E(t_1)$, $\aleph_I(t_1 * t_1) = \aleph_I(0) \leq \aleph_I(t_1)$, $\aleph_N(t_1 * t_1) = \aleph_N(0) \leq \aleph_N(t_1)$ and $S_E(t_1 * t_1) = S_E(0) \leq S_E(t_1)$, $S_I(t_1 * t_1) = S_I(0) \geq S_I(t_1)$, $S_N(t_1 * t_1) = S_N(0) \geq S_N(t_1)$. Suppose that $\aleph_E(7^{2q-1} t_1 * t_1) \geq \aleph_E(t_1)$, $\aleph_I(7^{2q-1} t_1 * t_1) \leq \aleph_I(t_1)$, $\aleph_N(7^{2q-1} t_1 * t_1) \leq \aleph_N(t_1)$ and $S_E(7^{2q-1} t_1 * t_1) \leq S_E(t_1)$, $S_I(7^{2q-1} t_1 * t_1) \geq S_I(t_1)$, $S_N(7^{2q-1} t_1 * t_1) \geq S_N(t_1)$. Then by assumption, $\aleph_E(7^{2(q+1)-1} t_1 * t_1) = \aleph_E(7^{2q+1} t_1 * t_1) = \aleph_E(7^{2q-1} t_1 * (t_1 * (t_1 * t_1))) = \aleph_E(7^{2q-1} t_1 * t_1) \geq \aleph_E(t_1)$, $\aleph_I(7^{2(q+1)-1} t_1 * t_1) = \aleph_I(7^{2q+1} t_1 * t_1) = \aleph_I(7^{2q-1} t_1 * (t_1 * (t_1 * t_1))) = \aleph_I(7^{2q-1} t_1 * t_1) \leq \aleph_I(t_1)$, $\aleph_N(7^{2(q+1)-1} t_1 * t_1) = \aleph_N(7^{2q+1} t_1 * t_1) = \aleph_N(7^{2q-1} t_1 * (t_1 * (t_1 * t_1))) = \aleph_N(7^{2q-1} t_1 * t_1) \leq \aleph_N(t_1)$ and $S_E(7^{2(q+1)-1} t_1 * t_1) = S_E(7^{2q+1} t_1 * t_1) = S_E(7^{2q-1} t_1 * (t_1 * (t_1 * t_1))) = S_E(7^{2q-1} t_1 * t_1) \leq S_E(t_1)$,

$S_I(7^{2(q+1)-1}t_1 * t_1) = S_I(7^{2q+1}t_1 * t_1) = S_I(7^{2q-1}t_1 * (t_1 * (t_1 * t_1))) = S_I(7^{2q-1}t_1 * t_1) \geq S_I(t_1)$, $S_N(7^{2(q+1)-1}t_1 * t_1) = S_N(7^{2q+1}t_1 * t_1) = S_N(7^{2q-1}t_1 * (t_1 * (t_1 * t_1))) = S_N(7^{2q-1}t_1 * t_1) \geq S_N(t_1)$, which prove (1),(2),(3),(4),(5) and (6), similarly we can prove the remaining cases (7) and (8).

Note: The sets denoted by $I_{\mathfrak{N}_{\Xi}}$ and $I_{S_{\Xi}}$ are also subalgebras of X , which are defined as: $I_{\mathfrak{N}_{\Xi}} = \{t_1 \in X | \mathfrak{N}_{\Xi}(t_1) = \mathfrak{N}_{\Xi}(0)\}$, $I_{S_{\Xi}} = \{t_1 \in X | S_{\Xi}(t_1) = S_{\Xi}(0)\}$.

Theorem 3.12 Let $\mathfrak{R} = (\mathfrak{N}_{\Xi}, S_{\Xi})$ be a NCMNSU of X . Then the sets $I_{\mathfrak{N}_{\Xi}}$ and $I_{S_{\Xi}}$ are subalgebras of X .

Proof. Let $t_1, t_2 \in I_{\mathfrak{N}_{\Xi}}$. Then $\mathfrak{N}_{\Xi}(t_1) = \mathfrak{N}_{\Xi}(0) = \mathfrak{N}_{\Xi}(t_2)$ and $\mathfrak{N}_{\Xi}(t_1 * t_2) \geq \min\{\mathfrak{N}_{\Xi}(t_1), \mathfrak{N}_{\Xi}(t_2)\} = \mathfrak{N}_{\Xi}(0)$. By using Proposition 2.3, we know that $\mathfrak{N}_{\Xi}(t_1 * t_2) = \mathfrak{N}_{\Xi}(0)$ or equivalently $t_1 * t_2 \in I_{\mathfrak{N}_{\Xi}}$.

Let $t_1, t_2 \in I_{S_{\Xi}}$. Then $S_{\Xi}(t_1) = S_{\Xi}(0) = S_{\Xi}(t_2)$ and $S_{\Xi}(t_1 * t_2) \leq \max\{S_{\Xi}(t_1), S_{\Xi}(t_2)\} = S_{\Xi}(0)$. Again by using Proposition 2.3, we know that $S_{\Xi}(t_1 * t_2) = S_{\Xi}(0)$ or equivalently $t_1 * t_2 \in I_{S_{\Xi}}$. Hence the sets $I_{\mathfrak{N}_{\Xi}}$ and $I_{S_{\Xi}}$ are subalgebras of X .

Theorem 3.13 Let B be a nonempty subset of X and $\mathfrak{R} = (\mathfrak{N}_{\Xi}, S_{\Xi})$ be a neutrosophic cubic set of X defined by

$$\mathfrak{N}_{\Xi}(t_1) = \begin{cases} [\kappa_{\Xi_1}, \kappa_{\Xi_2}], & \text{if } t_1 \in B \\ [\varphi_{\Xi_1}, \varphi_{\Xi_2}] & \text{otherwise,} \end{cases}, S_{\Xi}(t_1) = \begin{cases} \omega_{\Xi}, & \text{if } t_1 \in B \\ \varrho_{\Xi}, & \text{otherwise} \end{cases}$$

$\forall [\kappa_{\Xi_1}, \kappa_{\Xi_2}], [\varphi_{\Xi_1}, \varphi_{\Xi_2}] \in D[0,1]$ and $\omega_{\Xi}, \varrho_{\Xi} \in [0,1]$ with $[\kappa_{E_1}, \kappa_{E_2}] \geq [\varphi_{E_1}, \varphi_{E_2}]$, $[\kappa_{I_1}, \kappa_{I_2}] \leq [\varphi_{I_1}, \varphi_{I_2}]$, $[\kappa_{N_1}, \kappa_{N_2}] \leq [\varphi_{N_1}, \varphi_{N_2}]$, and $\omega_E \leq \varrho_E$, $\omega_I \geq \varrho_I$, $\omega_N \geq \varrho_N$. Then \mathfrak{R} is a NCMNSU of $X \Leftarrow B$ is a subalgebra of X . Moreover, $I_{\mathfrak{N}_{\Xi}} = B = I_{S_{\Xi}}$.

Proof. Let \mathfrak{R} be a NCMNSU of X and $t_1, t_2 \in X$ such that $t_1, t_2 \in B$. Then $\mathfrak{N}_E(t_1 * t_2) \geq \min\{\mathfrak{N}_E(t_1), \mathfrak{N}_E(t_2)\} = \min\{[\kappa_{E_1}, \kappa_{E_2}], [\kappa_{E_1}, \kappa_{E_2}]\} = [\kappa_{E_1}, \kappa_{E_2}]$, $\mathfrak{N}_I(t_1 * t_2) \leq \max\{\mathfrak{N}_I(t_1), \mathfrak{N}_I(t_2)\} = \max\{[\kappa_{I_1}, \kappa_{I_2}], [\kappa_{I_1}, \kappa_{I_2}]\} = [\kappa_{I_1}, \kappa_{I_2}]$, $\mathfrak{N}_N(t_1 * t_2) \leq \max\{\mathfrak{N}_N(t_1), \mathfrak{N}_N(t_2)\} = \max\{[\kappa_{N_1}, \kappa_{N_2}], [\kappa_{N_1}, \kappa_{N_2}]\} = [\kappa_{N_1}, \kappa_{N_2}]$ and $S_E(t_1 * t_2) \leq \max\{S_E(t_1), S_E(t_2)\} = \max\{\omega_E, \omega_E\} = \omega_E$, $S_I(t_1 * t_2) \geq \min\{S_I(t_1), S_I(t_2)\} = \min\{\omega_I, \omega_I\} = \omega_I$, $S_N(t_1 * t_2) \geq \min\{S_N(t_1), S_N(t_2)\} = \min\{\omega_N, \omega_N\} = \omega_N$. Therefore $t_1 * t_2 \in B$. Hence, B is a subalgebra of X . Conversely, suppose that B is a subalgebra of X and $t_1, t_2 \in X$. Consider two cases.

Case 1: If $t_1, t_2 \in B$ then $t_1 * t_2 \in B$, thus $\mathfrak{N}_E(t_1 * t_2) = [\kappa_{E_1}, \kappa_{E_2}] = \min\{\mathfrak{N}_E(t_1), \mathfrak{N}_E(t_2)\}$, $\mathfrak{N}_I(t_1 * t_2) = [\kappa_{I_1}, \kappa_{I_2}] = \max\{\mathfrak{N}_I(t_1), \mathfrak{N}_I(t_2)\}$, $\mathfrak{N}_N(t_1 * t_2) = [\kappa_{N_1}, \kappa_{N_2}] = \max\{\mathfrak{N}_N(t_1), \mathfrak{N}_N(t_2)\}$, and $S_E(t_1 * t_2) = \omega_E = \max\{S_E(t_1), S_E(t_2)\}$, $S_I(t_1 * t_2) = \omega_I = \min\{S_I(t_1), S_I(t_2)\}$, $S_N(t_1 * t_2) = \omega_N = \min\{S_N(t_1), S_N(t_2)\}$.

Case 2: If $t_1 \notin B$ or $t_2 \notin B$, then $\mathfrak{N}_E(t_1 * t_2) \geq [\varphi_{E_1}, \varphi_{E_2}] = \min\{\mathfrak{N}_E(t_1), \mathfrak{N}_E(t_2)\}$, $\mathfrak{N}_I(t_1 * t_2) \leq [\varphi_{I_1}, \varphi_{I_2}] = \max\{\mathfrak{N}_I(t_1), \mathfrak{N}_I(t_2)\}$, $\mathfrak{N}_N(t_1 * t_2) \leq [\varphi_{N_1}, \varphi_{N_2}] = \min\{\mathfrak{N}_N(t_1), \mathfrak{N}_N(t_2)\}$, and $S_E(t_1 * t_2) \leq \varrho_E = \max\{S_E(t_1), S_E(t_2)\}$, $S_I(t_1 * t_2) \geq \varrho_I = \min\{S_I(t_1), S_I(t_2)\}$, $S_N(t_1 * t_2) \geq \varrho_N = \min\{S_N(t_1), S_N(t_2)\}$. Hence \mathfrak{R} is a NCMNSU of X . Now, $I_{\mathfrak{N}_{\Xi}} = \{t_1 \in X, \mathfrak{N}_{\Xi}(t_1) = \mathfrak{N}_{\Xi}(0)\} = \{t_1 \in X, \mathfrak{N}_{\Xi}(t_1) = [\kappa_{\Xi_1}, \kappa_{\Xi_2}]\} = B$, and $I_{S_{\Xi}} = \{t_1 \in X, S_{\Xi}(t_1) = S_{\Xi}(0)\} = \{t_1 \in X, S_{\Xi}(t_1) = \omega_{\Xi}\} = B$.

Theorem 3.14 Let $\mathfrak{R} = (\mathfrak{N}_{\Xi}, S_{\Xi})$ be a neutrosophic cubic set of X . For $[s_{E_1}, s_{E_2}], [s_{I_1}, s_{I_2}], [s_{N_1}, s_{N_2}] \in D[0,1]$ and $t_{E_1}, t_{I_1}, t_{N_1} \in [0,1]$, the set $U(\mathfrak{N}_{\Xi} | ([s_{E_1}, s_{E_2}], [s_{I_1}, s_{I_2}], [s_{N_1}, s_{N_2}])) = \{t_1 \in X | \mathfrak{N}_E(t_1) \geq [s_{E_1}, s_{E_2}], \mathfrak{N}_I(t_1) \leq [s_{I_1}, s_{I_2}], \mathfrak{N}_N(t_1) \leq [s_{N_1}, s_{N_2}]\}$ is called upper $([s_{E_1}, s_{E_2}], [s_{I_1}, s_{I_2}], [s_{N_1}, s_{N_2}])$ -level of \mathfrak{R} and $L(S_{\Xi} | (t_{E_1}, t_{I_1}, t_{N_1})) = \{t_1 \in X | S_E(t_1) \leq t_{E_1}, S_I(t_1) \geq t_{I_1}, S_N(t_1) \geq t_{N_1}\}$ is called lower $(t_{E_1}, t_{I_1}, t_{N_1})$ -level of \mathfrak{R} . If $\mathfrak{R} = (\mathfrak{N}_{\Xi}, S_{\Xi})$ is NCMNSU of X , then the upper $[s_{E_1}, s_{E_2}]$ -level and lower t_{E_1} -level of \mathfrak{R} are subalgebras of X .

Proof. Let $t_1, t_2 \in U(\mathfrak{N}_{\Xi} | [s_{\Xi_1}, s_{\Xi_2}])$. Then $\mathfrak{N}_E(t_1) \geq [s_{E_1}, s_{E_2}]$ and $\mathfrak{N}_E(t_2) \geq [s_{E_1}, s_{E_2}]$. It follows that $\mathfrak{N}_E(t_1 * t_2) \geq \text{rmin}\{\mathfrak{N}_E(t_1), \mathfrak{N}_E(t_2)\} \geq [s_{E_1}, s_{E_2}] \Rightarrow t_1 * t_2 \in U(\mathfrak{N}_E | [s_{E_1}, s_{E_2}])$, $\mathfrak{N}_I(t_1) \leq [s_{I_1}, s_{I_2}]$ and $\mathfrak{N}_I(t_2) \leq [s_{I_1}, s_{I_2}]$. It follows that $\mathfrak{N}_I(t_1 * t_2) \leq \text{rmax}\{\mathfrak{N}_I(t_1), \mathfrak{N}_I(t_2)\} \leq [s_{I_1}, s_{I_2}] \Rightarrow t_1 * t_2 \in U(\mathfrak{N}_I | [s_{I_1}, s_{I_2}])$, $\mathfrak{N}_N(t_1) \leq [s_{N_1}, s_{N_2}]$ and $\mathfrak{N}_N(t_2) \leq [s_{N_1}, s_{N_2}]$. It follows that $\mathfrak{N}_N(t_1 * t_2) \leq \text{rmax}\{\mathfrak{N}_N(t_1), \mathfrak{N}_N(t_2)\} \leq [s_{N_1}, s_{N_2}] \Rightarrow t_1 * t_2 \in U(\mathfrak{N}_N | [s_{N_1}, s_{N_2}])$. Hence, $U(\mathfrak{N}_{\Xi} | [s_{\Xi_1}, s_{\Xi_2}])$ is a subalgebra of X . Let $t_1, t_2 \in L(S_{\Xi} | t_{\Xi_1})$. Then $S_E(t_1) \leq t_{E_1}$ and $S_E(t_2) \leq t_{E_1}$. It follows that $S_E(t_1 * t_2) \leq \max\{S_E(t_1), S_E(t_2)\} \leq t_{E_1} \Rightarrow t_1 * t_2 \in L(S_E | t_{E_1})$, $S_I(t_1) \geq t_{I_1}$ and $S_I(t_2) \geq t_{I_1}$. It follows that $S_I(t_1 * t_2) \geq \min\{S_I(t_1), S_I(t_2)\} \geq t_{I_1} \Rightarrow t_1 * t_2 \in L(S_I | t_{I_1})$, $S_N(t_1) \geq t_{N_1}$ and $S_N(t_2) \geq t_{N_1}$. It follows that $S_N(t_1 * t_2) \geq \min\{S_N(t_1), S_N(t_2)\} \geq t_{N_1} \Rightarrow t_1 * t_2 \in L(S_N | t_{N_1})$. Hence $L(S_{\Xi} | t_{\Xi_1})$ is a subalgebra of X .

Theorem 3.15 Let $\mathfrak{R} = (\mathfrak{N}_{\Xi}, S_{\Xi})$ is NCMNSU of X . Then $\mathfrak{N}([s_{\Xi_1}, s_{\Xi_2}]; t_{\Xi_1}) = U(\mathfrak{N}_{\Xi} | [s_{\Xi_1}, s_{\Xi_2}]) \cap L(S_{\Xi} | t_{\Xi_1}) = \{t_1 \in X | \mathfrak{N}_E(t_1) \geq [s_{E_1}, s_{E_2}], \mathfrak{N}_I(t_1) \leq [s_{I_1}, s_{I_2}], \mathfrak{N}_N(t_1) \leq [s_{N_1}, s_{N_2}], S_E(t_1) \leq t_{E_1}, S_I(t_1) \geq t_{I_1}, S_N(t_1) \geq t_{N_1}\}$ is a subalgebra of X .

Proof. This theorem can be proved by using Theorem 3.14. The converse of Theorem 3.15 is not valid, for which we present the example.

Example 3.2 Let $X = \{0, t_1, t_2, t_3, t_4, t_5\}$ be a BF-algebra used in above example and $\mathfrak{R} = (\mathfrak{N}_{\Xi}, S_{\Xi})$ is a neutrosophic cubic set defined by

	0	t_1	t_2	t_3	t_4	t_5
\mathfrak{N}_T	[0.5,0.7]	[0.6,0.7]	[0.6,0.7]	[0.2,0.3]	[0.4,0.5]	[0.4,0.5]
\mathfrak{N}_I	[0.4,0.6]	[0.5,0.6]	[0.5,0.6]	[0.5,0.7]	[0.4,0.4]	[0.4,0.8]
\mathfrak{N}_F	[0.3,0.5]	[0.3,0.6]	[0.3,0.6]	[0.3,0.6]	[0.2,0.3]	[0.2,0.3]

	0	t_1	t_2	t_3	t_4	t_5
S_T	0.2	0.4	0.4	0.6	0.4	0.6
S_I	0.3	0.5	0.5	0.7	0.5	0.7
S_F	0.4	0.6	0.6	0.8	0.6	0.8

Now $\mathfrak{N}([s_{\Xi_1}, s_{\Xi_2}]; t_{\Xi_1}) = U(\mathfrak{N}_{\Xi} | [s_{\Xi_1}, s_{\Xi_2}]) \cap L(S_{\Xi} | t_{\Xi_1}) = \{t_1 \in X | \mathfrak{N}_T(t_1) \geq [s_{T_1}, s_{T_1}], S_T(t_1) \leq t_T\} = \{0, t_1, t_3\} \cap \{0, t_1, t_2, t_3\} = \{0, t_1, t_3\}$ is a subalgebra of X , similarly we can find this for indeterminate and non membership elements. But $\mathfrak{R} = (\mathfrak{N}_{\Xi}, S_{\Xi})$ is not a neutrosophic cubic subalgebra, since $\mathfrak{N}_T(t_1 * t_4) = [0.2, 0.3] \not\geq [0.4, 0.5] = \text{rmin}\{\mathfrak{N}_T(t_1), \mathfrak{N}_T(t_4)\}$, $\mathfrak{N}_I(t_1 * t_2) = [0.4, 0.8] \not\leq [0.4, 0.6] = \text{rmax}\{\mathfrak{N}_I(t_1), \mathfrak{N}_I(t_2)\}$, similarly we can find this for non membership element and $\lambda_T(a_2 * a_4) = 0.6 \not\leq 0.4 = \max\{\lambda_T(a_2), \lambda_T(a_4)\}$, similarly we can find this for indeterminate and non membership elements.

4. CONCLUSION

In this paper, neutrosophic cubic MN-subalgebra was introduced and its few helpful results and new characteristics were studied. The investigation of this new sort of subalgebra will help analysts to apply this subalgebra on various algebras. We are recommending some ideas like multiplication, and cartesian product to apply this work.

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The Special Neutrosophic Functions

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Abstract

In this study, we introduce the notion of special neutrosophic functions as new kinds of neutrosophic function defined in a neutrosophic logic. As particular cases, we present the notions of neutrosophic Floor (greatest integer), neutrosophic Absolute Function and neutrosophic Signum Function. Moreover, we draw its neutrosophic graph representation and discuss similarities and differences for these special neutrosophic functions between the classic case and neutrosophic case. We investigate some properties and prove them. However, we often need the definition of absolute value function, especially in the metric space. Therefore, we introduce its initial definition in this study.

Keywords: Neutrosophic relation, Neutrosophic function, Neutrosophic derivative, Neutrosophic integral Neutrosophic representation.

1. Introduction

In our life, there is three main types of logic. The first one is the classical logic which has two values, ‘true or false’, ‘0 or 1’. The second one is the fuzzy logic which was first introduced by Dr. Lotfi Zadeh in 1960s. It has more than two values. This means that it has more than ‘true or false’ because they are considered simple in this type of logic. With fuzzy logic, propositions can be represented with degrees of truth and falseness [1, 2, 3]. The final type of logic is the neutrosophic logic which is an extension of the fuzzy logic in which indeterminacy I is included taking into consideration ($I^n = nI = I, \forall n \in N$) [4, 5]. This idea has inspired a lot of researchers and opened up a wide range of scientific research in many ways.

Due to the importance of calculus, Florentin Smarandache presented the basic of Neutrosophic Pre-calculus and Neutrosophic Calculus, which studies the neutrosophic functions [6, 7]. A neutrosophic Function $Nf : D \longrightarrow R$ is a function, which has some indeterminacy, with respect to its domain of definition, to its range, or to the relation that connect between elements in D with elements in R . Especially; he also defined the neutrosophic exponential function and neutrosophic logarithmic function.

The idea of the perception of pentagonal neutrosophic number from different aspects and the score function in pentagonal neutrosophic domain was introduced in [8,9]. Additionally, in [10,11,12,13,14,15,16] the single neutrosophic value and its properties of different kinds have been identified. Moreover, a lot of algebraic neutrosophic structures have been identified, such as neutrosophic R-modules [17,18] and also in the area of neutrosophic topological space [19,20].

2. Preliminary

In this section, we present the basic definitions that are useful in this research.

2.1 Neutrosophic Subset Relation[6]:

A Neutrosophic Subset Relation β , between two sets A and B , is a set of ordered pairs of the form (A', B') , where A' is a subset of A , and B' a subset of B , with some indeterminacy. A neutrosophic relation β , besides sure ordered pairs (A', B') that 100% belong to β , can also contain potential ordered pairs (A'', B'') , where A'' is a subset of A , and B'' a subset of B , which may be possible to belong to β , but it is unknown in what degree, or that partially belong to β with the neutrosophic value (T, I, F) where T means degree of appurtenance to β , I means degree of indeterminate appurtenance, and F means degree of non-appurtenance.

2.2 Neutrosophic Functions[6]:

A Neutrosophic Function is a neutrosophic relation in which the vertical line test does not necessarily work. However, in this case, the neutrosophic function coincides with the neutrosophic relation. Generally, a neutrosophic function is a function that has some indeterminacy [with respect to one or more of its formula, domain, or range].

Example [6]: Let's we have $f : \{1, 2, 3\} \rightarrow \{a, b, c, d\}$ is a neutrosophic function defined as: $f(1) = a$, $f(2) = b$, but $f(3) = c \text{ or } d$ [we are not sure], so we can write $f(3) = I$. If we consider a neutrosophic diagram representation of this neutrosophic function, we have:

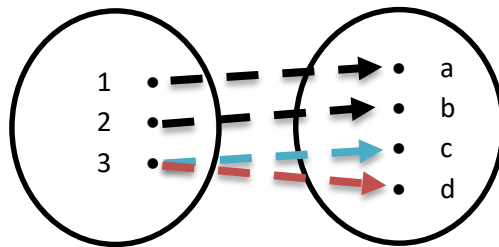
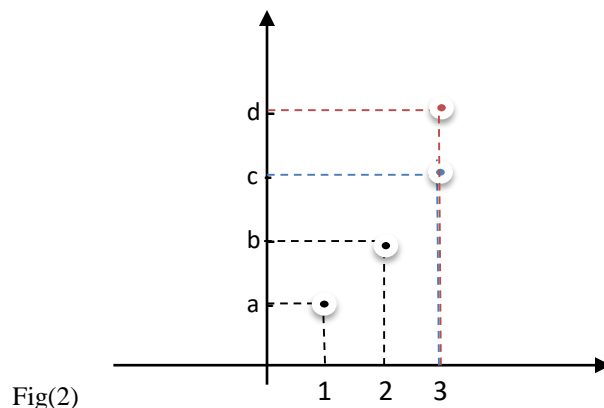


Fig (1)

The color arrows mean that we are not sure if the element 3 is connected to the element c , or if 3 is connected to d . Similarly, for a graph representation:



Fig(2)

This example can be rephrased in another way, that 3 is connected with d only partially, let's say $(3, d)_{(0.6, 0.2, 0.5)}$ which means that 60% the 3 is connected with d , 20% . It is not clear whether it is connected or unconnected, and 50% the 3 is not connected with d . The sum of components $0.6+0.2+0.5=1.3$ is more than 1 because the three sources providing information about connection, indeterminacy, non-connection respectively are independent and use different criteria of evaluation.

As we see, this neutrosophic function is neither a function nor a relation in the classical case.

Example: [6] Let's consider $h: R \rightarrow R$ a different type of neutrosophic function defined as: $\forall x \in R, h(x) \in [2, 3]$, so we can write $h(x) = I$. Therefore, we just know that this function is bounded by the horizontal lines $y = 2$ and $y = 3$.

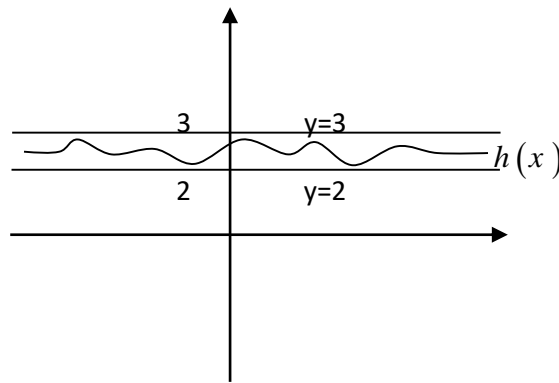


Fig (3)

We can modify $h(x)$ and get a constant neutrosophic function (or thick function): $l: R \rightarrow P(R)$ defined as: $\forall x \in R, l(x) = [2, 3]$ Where $P(R)$ is the set of all subsets of R .

For example, is the vertical segment of line $[2, 3]$.

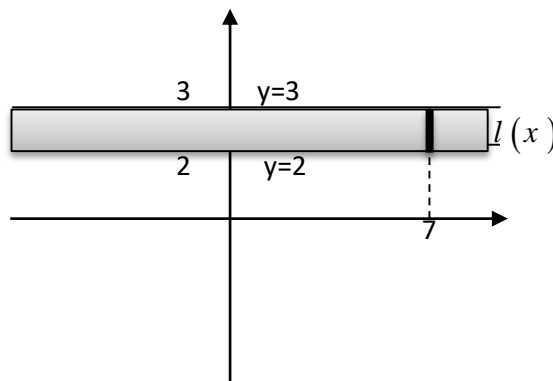


Fig (4)

Example:[6] A non-constant neutrosophic thick function: $k: R \rightarrow P(R)$ defined as: $\forall x \in R, k(x) = [2x, 2x + 1]$ whose neutrosophic representation is:

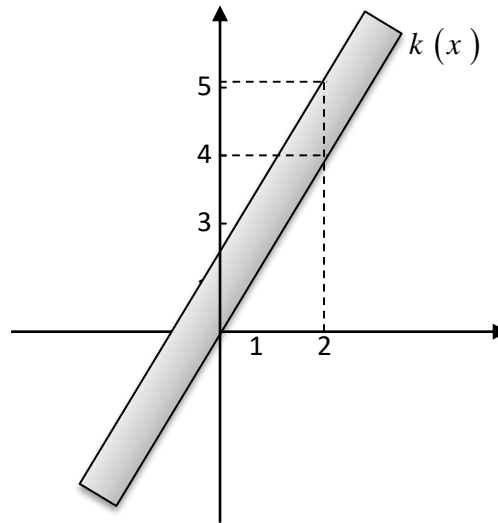


Fig (5)

2.3 Neutrosophic Derivative[6] :

The general definition of the neutrosophic derivative of function $f'_N(x)$ is:

$$f'_N(x) = \lim_{h \rightarrow 0} \frac{[\inf f(X+h) - \inf f(X), \sup f(X+h) - \sup f(X)]}{h}$$

Example: Let's $f: R \rightarrow R \cup \{I\}$ a neutrosophic function defined as: $f_N(x) = [x^2 + 2x, x^3]$ then :

$$\begin{aligned} f'_N(X) &= \lim_{h \rightarrow 0} \frac{[(x+h)^2 + 2(x+h) - x^2 - 2x, (x+h)^3 - x^3]}{h} \\ &= \left[\lim_{h \rightarrow 0} \frac{(x+h)^2 + 2(x+h) - x^2 - 2x}{h}, \lim_{h \rightarrow 0} \frac{(x+h)^3 - x^3}{h} \right] \\ &= \left[\frac{d}{dx}(x^2 + 2x), \frac{d}{dx}x^3 \right] \\ &= [2x + 2, 3x^2] \end{aligned}$$

Example: Let $f: R \rightarrow R \cup \{I\}$ a neutrosophic function defined as: $f_N(x) = 3x - x^2I$ then :

$$\begin{aligned} f'_N(x) &= \lim_{h \rightarrow 0} \frac{f_N(x+h) - f_N(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{[3(x+h) - (x+h)^2I] - [3x - x^2I]}{h} \\ &= \lim_{h \rightarrow 0} \frac{h \cdot (3 - 2xI - hI)}{h} = 3 - 2xI - 0 \cdot I = 3 - 2xI \end{aligned}$$

2.4 Neutrosophic Integral [7]


Using the neutrosophic measure, we will define a neutrosophic integral. The neutrosophic integral of a function

Nf is written as: $\int ({}_X Nf) dv$

Where X is a neutrosophic measure space, and also the integral is taken with respect to the neutrosophic measure ν . Indeterminacy related to integration can occur in various ways: with respect to the value of the integrated function, with respect to the lower or upper limit of integration, or with respect to the space and its measure.

Example: Let $f : R \rightarrow R \cup \{I\}$ a neutrosophic function defined as: $f(x) = 2x^3 + (x^2 + 3)I$ then :

$$\begin{aligned} F(x) &= \int [4x^3 + (x^2 + 3)I] dx \\ &= \int 4x^3 dx + \int [(x^2 + 3)I] dx \\ &= x^4 + \frac{x^3}{3}I + 3xI + c \end{aligned}$$


 Integration neutrosophic Constant
 $c = a + bI : a, b \in R$

3. The Special Neutrosophic Functions:

3.1 Piecewise function:

A Neutrosophic piecewise Function is a piecewise function that has some indeterminacy [with respect to one or more of: its domain, formula, or range].

The Neutrosophic piecewise function is may not be a classical function in general. However, we can say when indeterminacy doesn't exist we will be back the classical case again.

Example: Let's consider a neutrosophic piecewise function which has indeterminacy with respect to its domain:

$$f_1(x) = \begin{cases} x^2 & | x \notin \{-1, 1\} \\ [2, 3] & | x = -1 \text{ or } 1 \end{cases}$$

It's clear that is $f_1(-1) \neq f_1(1) \neq 1$ and $f(I) = [2, 3]$

As in the classical way we can draw the neutrosophic graph:

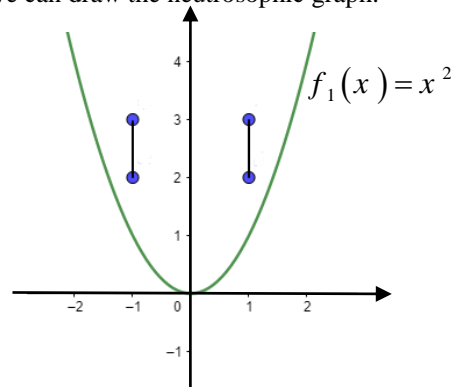


Fig (6)

Example: Let's consider a neutrosophic piecewise function, which has indeterminacy with respect to its formula:

$$f_2(x) = \begin{cases} [2x + 1, 6x] & | x \neq 0 \\ [1, 3] & | x = 0 \end{cases}$$

It's clear that $f_2(x) = I : x \neq 0$ and $f_2(0) = [1, 3]$

As in the classical way, we can draw the neutrosophic graph:

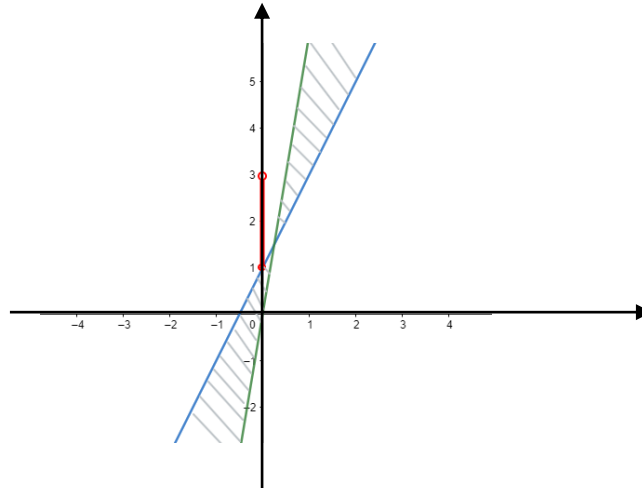


Fig (7)

Example: Let's consider a neutrosophic piecewise function, which has indeterminacy with respect to its range:

$$f_3(x) = \begin{cases} \frac{1}{x-5} & | x \neq 5 \\ 2 \text{ or } 4 & | x = 5 \end{cases}$$

It's clear that $f_3(5) = I$ and $f_3(x) = \frac{1}{x-5} : x \neq 5$

As in the classical way, we can draw the neutrosophic graph:

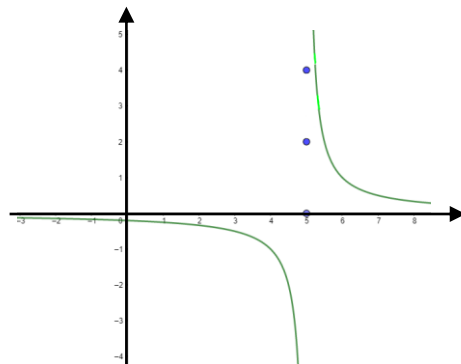


Fig (8)

3.2 Signum function:

A Neutrosophic signum Function ($N \operatorname{sgn}$) is a signum function which has some indeterminacy [with respect to one or more of: its domain, formula, or range] in two ways as follows:

$$1) N \operatorname{sgn}(x + I) = \begin{cases} 1 : x > 0 \text{ and } I = 0 \\ 0 : x = 0 \text{ and } I \neq 0 \\ -1 : x < 0 \text{ and } I = 0 \end{cases}$$

$$2) N \operatorname{sgn}(x) = \begin{cases} 1 : x > 0 \\ 0 + I : x = 0 \\ -1 : x < 0 \end{cases}$$

The indeterminacy here is suitable to the problem conditions.

A Neutrosophic signum Function may be continuous at (0) due to the indeterminacy in contrary to the classical case.

Example: Let's consider a neutrosophic signum function, which has indeterminacy with respect to its domain:

$$N \operatorname{sgn}(x - 3 + 2I) = \begin{cases} 1 : x > 3 \text{ and } I = 0 \\ 0 : x = 3 \text{ and } I \neq 0 \\ -1 : x < 3 \text{ and } I = 0 \end{cases}$$

As in the classical way, we can draw the neutrosophic graph:

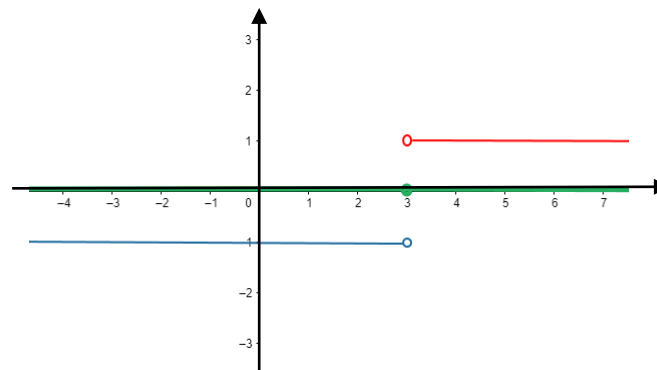


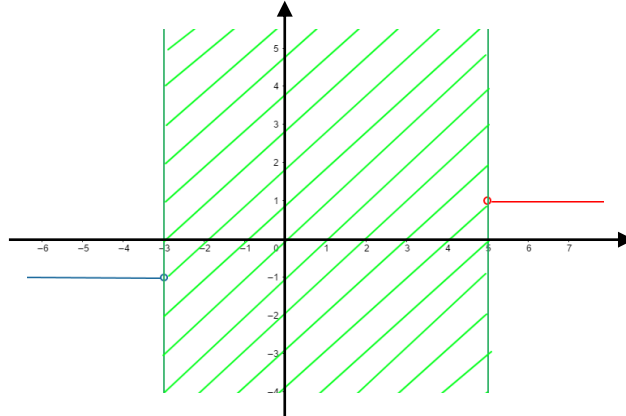
Fig (9)

From the graph, we notice that the Neutrosophic signum function is continuous at 3 in the contrary to the classical case, and when there is no indeterminacy the green color will fade. Therefore, we will go back to the classical case.

Example: Let's consider a neutrosophic signum function which has indeterminacy with respect to its formula and range:

$$N \operatorname{sgn}(x+2) = \begin{cases} 1 : x > 5 \\ 0+I : x = -2+I \\ -1 : x < -3 \end{cases}$$

As in the classical way we can draw the neutrosophic graph:



Fig(10)

Notice what the indeterminacy has made in the graph. It becomes like a spectrum around zero.

3.3 Neutrosophic Absolute Function

A Neutrosophic Absolute Function ($Nabs$) is an Absolute Function which has some indeterminacy [with respect to one or more of: its domain, formula, or range] in three ways as follows:

$$1) Nabs(x+I) = \begin{cases} x : x > 0 \text{ and } I = 0 \\ 0+I : x = 0 \text{ and } I \neq 0 \\ -x : x < 0 \text{ and } I = 0 \end{cases}$$

$$2) Nabs(x) = \begin{cases} x : x > 0 \\ 0+I : x = 0 \\ -x : x < 0 \end{cases}$$

$$3) Nabs(x+I) = \begin{cases} x+I : x > 0 \\ 0+I : x = 0 \\ -x+I : x < 0 \end{cases}$$

Properties:

$$1) Nabs(x + I) = abs(x) + I$$

$$2) Nabs(0 + I) = abs(0) + I = I$$

$$3) Nabs(x + I) + I = abs(x) + 2I = Nabs(x + I)$$

$$4) Nabs(x + y + I) \leq Nabs(x + I) + Nabs(y + I)$$

proof :

$$\begin{aligned} Nabs(x + y + I) &= abs(x + y) + I \\ &\leq abs(x) + abs(y) + I \\ &= abs(x) + I + abs(y) + I \\ &= Nabs(x + I) + Nabs(y + I) \end{aligned}$$

$$5) Nabs(x + I) = y + I$$

$$\Rightarrow abs(x) + I = y + I$$

$$\Rightarrow abs(x) = y + I \Rightarrow \begin{cases} x = y + I \\ x = -y + I \end{cases}$$

Example: Let's consider a neutrosophic Absolute function which has indeterminacy with respect to its domain:

$$Nabs(x + 3I) = \begin{cases} x : x > 0 \text{ and } I = 0 \\ 0 + I : x = 0 \text{ and } I \neq 0 \\ -x : x < 0 \text{ and } I = 0 \end{cases}$$

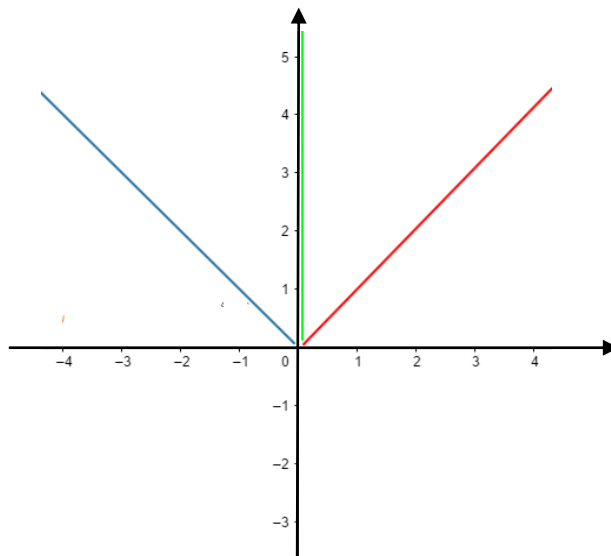


Fig (11)

Example: Let's consider a neutrosophic Absolute function which has indeterminacy with respect to its formula, and range:

$$Nabs(x-2) = \begin{cases} x-2 : x > 4 \\ 0+I : x \in [1, 4] \\ 2-x : x < 1 \end{cases}$$

As in the classical way, we can draw the neutrosophic graph:

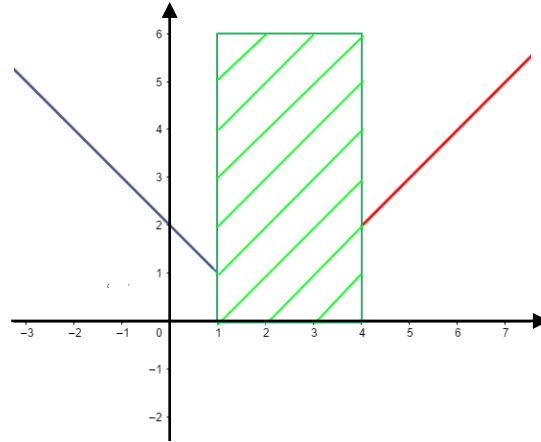


Fig (12)

3.4 Neutrosophic Floor (greatest integer) Function

A Neutrosophic Floor (greatest integer) ($Nfloor[\cdot]$) is a floor (greatest integer) that has some indeterminacy [with respect to one or more of: its domain, formula, or range] in two ways as follows:

$$1) Nfloor[\cdot] = \begin{cases} \cdot \\ \cdot \\ -1 : -1 \leq x + I < 0 \\ 0 : 0 \leq x + I < 1 \\ 1 : 1 \leq x + I < 2 \\ \cdot \\ \cdot \end{cases}$$

As in the classical way, we can draw the neutrosophic graph:

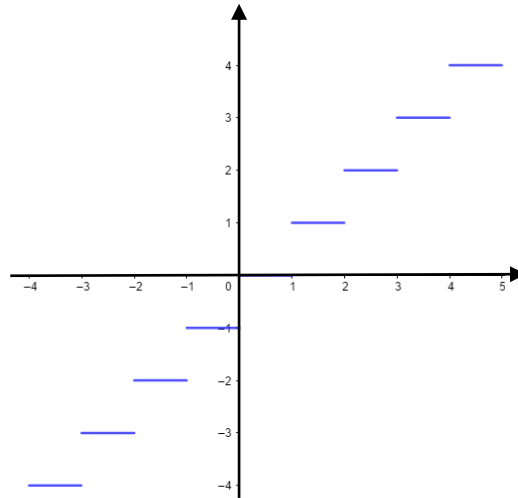


Fig (13)

Here we can say that there is no difference between the neutrosophic case and the classical case.

$$2)Nf\text{loor}[\![x]\!] = \begin{cases} \cdot \\ \cdot \\ -1+I & : -1 \leq x < 0 \\ 0+I & : 0 \leq x < 1 \\ 1+I & : 1 \leq x < 2 \\ \cdot \\ \cdot \end{cases}$$

As in the classical way, we can draw the neutrosophic graph:

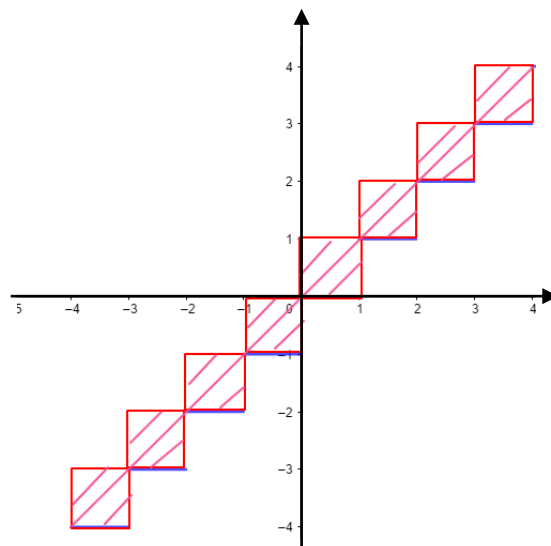


Fig (14)

Properties:

$$1) N\text{floor}[\![x + I]\!] = [\![x]\!] + I = x + a + I : 0 \leq a < 1$$

$$2) \forall n \in N(I)$$

$$\begin{aligned} N\text{floor}[\![n + I]\!] &= [\![n]\!] + I \\ &= n + I \end{aligned}$$

$$3) N\text{floor}[\![x + y + I]\!] \geq N\text{floor}[\![x + I]\!] + N\text{floor}[\![y + I]\!]$$

proof :

$$\begin{aligned} N\text{floor}[\![x + y + I]\!] &= [\![x + y]\!] + I \\ &\geq [\![x]\!] + [\![y]\!] + I \\ &= [\![x]\!] + I + [\![y]\!] + I \\ &= N\text{floor}[\![x + I]\!] + N\text{floor}[\![y + I]\!] \end{aligned}$$

$$4) \forall n \in N(I)$$

$$N\text{floor}[\![x + n + I]\!] = N\text{floor}[\![x + I]\!] + n$$

proof :

$$\begin{aligned} N\text{floor}[\![x + n + I]\!] &= [\![x + n]\!] + I \\ &= [\![x]\!] + n + I \\ &= [\![x]\!] + I + n \\ &= N\text{floor}[\![x + I]\!] + n \end{aligned}$$

4. Conclusions

In this research, we firstly obtained new kinds of neutrosophic functions and focused on the Neutrosophic representation and proved some properties. In addition, we showed that the neutrosophic functions is not a function in the classical case, but in some especial cases there were an coincidence between the neutrosophic case and the classical case.

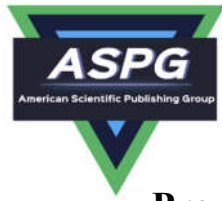
5. Future Research Directions

As a future work, this article can be extended to include continuity and derivation and integration as well as the definition and applications of the Neutrosophic Cartesian.

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Proposal for a new theory of neutrosophic application of the evidence in the Ecuadorian criminal process

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Abstract

In this article, the main objective was to examine the articulation mechanism of the guiding principles of evidentiary law, the backbone of the criminal procedure directed at judges so as not to make inexcusable mistakes. A new theory called reasoned equivalence based on numerical neutrosophics by considering each evidentiary principle <A> along with its opposite or negation <Anti-A> and the spectrum of neutralities <Neut-A>. The data collection techniques responded to participant observation and the Delphi technique, after having gathered the opinion of 60 collaborating criminal lawyers about the problem through the exercise of the profession. The construction of the instrument fell to an observation guide. The results gave the judicial practice a marked formative value, by establishing relationships between the content of the evidence and the development of oral litigation techniques aimed at the promotion and evacuation of evidence to contribute to a certain criminal process, the evidence necessary to that the judge can come to the knowledge and conviction of the procedural truth of the facts.

Keywords: Criminal; equivalence reasoned; evidence; method; neutrosophic numbers; process

1. Introduction

Proving in its broadest and most contemporary expression tautologically means convincing the judge about the certainty of the existence of an event. Besides, it is constituting a legitimate and open reaffirmation of the probationary right. In most criminal cases it is affirmed that proof is the verification of something; the truth about a fact. Criminal evidence is the circumstances submitted to the judge for his judgment. Therefore, it shows the veracity of what is alleged about the facts in a trial [1], [2].

It is very important to deepen the knowledge of what concerns the presentation of evidence in a criminal procedure. The function of the test, in general terms, supports: (...) the obtaining of the truth (...). The material truth of the facts would reside in total knowledge of them by the judge [3]. That said, the emphasis should be put especially in the interrogation, including recognition of places, people, or things as well as proof of judicial inspection, since this gives validity to the criminal process. Therefore, it is necessary that the criminal procedures meet impermissible minimum requirements and it is not illegitimate to avoid the nullity of the act or the whole process. As a consequence, it would make it very difficult for the Judge to arrive at the truth of the facts; the purpose that pursues all investigation in the criminal process [4], [5], [6].

Any consideration that can be made regarding the subject under study is relatively complex given the needs that currently arise when accessing justice, therefore some factors must be overcome to have effective, transparent justice and expeditiously, and thus, fulfill the mission that all legal professionals have, the defenders, prosecutors of the Public Prosecutor's Office, criminal lawyers in practice and officials of the Judicial Power [7].

Next, the considerations after the transformation of the object of study, the following variables arise in chronological order: (i) the statement of the accused; (ii) the interrogation or examination; (iii) cross-examination or contrary-examination; (iv) the test of judicial inspection *ex-ante* ocular inspection; (v) the recognition of people, things, places and; (vi) the proof evaluation system.

The investigation motivates us to recognize the absolute independence of the Administration of Justice. It requires that the power of attorney exercise the power on itself, without the interference of another power that displays unlimited faculties on the justice operator, much more if this same organism who designates it, appoints and swears it, to finally sanction it disciplinary way proceeding for “inexcusable error” considered in Article 109, Number 7, of the Organic Code of the Judicial Function [8]; and not only of the internal independence which was submitted; but the same external independence is subject to interference that the political power does on the justice administration call by this legal figure that has not had a sustainable and satisfactory legal explanation.

The fundamental contribution of the study focuses on the demonstration of the importance of the evidence right, the backlist of the criminal process, specifically from the moment the arrest of the alleged suspect arises based on the principles of general interest, the social order of the freedom of the accused, protected on a level of equality in opportunities.

The problem is determined by the assumptions where the vice of the illegal evidence appears during the criminal processes, to search through the analysis of the same solution that offers the appropriate procedural corrective.

In this sense, it holds in the indicated cases that, in the Ecuadorian criminal process, sometimes the judge excludes evidence without vices of illegality at violating the principle of probation [6].

Successive and contemporaneously, concerning the evidence obtained illicitly: The judge at the time of assessing the evidence produced a trial must analyze with great care, firstly, if the proof does not suffer from the illegality, that is, contrary to a constitutional or legal rule [9].

Based on the arguments presented, empowered in the evidentiary dimension, the general educational objective at a critical-transferential level was circumscribed in examine the purpose of the judicial evidence in the criminal process, the means and, the probative sources based on the general principles of legality and legitimacy of the criminal evidence.

2. Problem formulation

The problem is determined by the assumptions where the vice of the criminal evidence appears in the course of the processes, to search through the analysis of the same solution that offers the appropriate procedural corrective to each of these processes; pursuing the understanding and scope of the evidentiary mechanisms for the factual determination of reality in the construction of judicial evidence, in correspondence with the following hypothesis:

How is the legality and legitimacy of the criminal evidence articulated for probation?

3. Neutrosophic numbers of unique value to represent the jurisdiction in the criminal procedural field

The definition of truth value in neutrosophic logic is represented as $N = \{(T, I, F) : T, I, F \subseteq [0, 1]\}$ n , representing a neutrosophic valuation [10], [11]. Specifically, one of the mathematical theories that generalize the classical and fuzzy theories is the demonstration of statistical hypotheses, which is used in the present study [12], [13]. It is considered as a mapping of a group of propositional formulas to N , and for each sentence, to obtain the result through the following expression.

$$v(p) = (T, I, F)$$

Starting from U that represents the universe of discourse and the neutrosophic set $Ie \subset U$.

Where:

Ie is formed by the set of evaluative indicators that define a legal jurisdiction.

It should be noted that the following triads are used in legal Sciences: $\langle A \rangle$ be a physical entity (i.e. concept, notion, object, space, field, idea, law, property, state, attribute, theorem, theory), $\langle \text{anti}A \rangle$ be the opposite of $\langle A \rangle$, and $\langle \text{neut}A \rangle$ be their neutral (i.e. neither $\langle A \rangle$ nor $\langle \text{anti}A \rangle$, but in between) [14].

In the physical field, formal logic operates as a “Paradoxist Physics Neutrosophic Physics is an extension of Paradoxist Physics, since Paradoxist Physics is a combination of physical contradictories $\langle A \rangle$ and $\langle \text{anti}A \rangle$ only that hold together, without referring to their neutrality $\langle \text{neut}A \rangle$. Paradoxist Physics describes collections of objects or states that are individually characterized by contradictory properties, or are characterized neither by a property nor by the opposite of that property, or are composed of contradictory sub-elements. Such objects or states are called paradoxist entities”. [14].

Let $T(x)$, $I(x)$, $F(x)$ be the functions that describe the degrees of correlation or non-correlation, respectively, of a generic element $x \in U$, concerning the neutrosophic set Ie .

Therefore, when considering the clear (classic) principle of legality and legitimacy of criminal evidence. Yes only if the criminal procedural law, it is equivalent to excluding the illegality of the evidence by refuting it as exclusionary when it is qualified by the judge as pertinent to the criminal process by recognizing that in criminal trials in Ecuador there is evidentiary freedom, therefore it is valid to affirm that in all In criminal trials, there is a 100% evaluation of the criminal evidence by the judge.

Using the notation of neutrophilic numbers, we write that in Ecuador there is (1, 0, 0) probation, which means that country is 100% legal, 0% undetermined legal, and 0% illegal.

However, the investigation shows that some courts exclude the validity of criminal evidence, invoking aspects such as impertinence of the evidence, misusing evidentiary law. Therefore, it is determined that probation is among the said in proportion to a fifth (20%) excluding equivalent to (0.8, 0, 0.2) - freedom and probative legitimacy [15].

4. Problem solution

Within the framework of rational choice, microanalysis is carried out based on a dual result that reveals open and axial coding (Andréu Abela et. al, 2007) [16] oriented to find the signifier of the data.

The open coding consisted of the analytical procedure employing which the data were delimited giving way to the thoughts, ideas, and meanings that contain it, to discover, catalog and develop concepts to arrive at a new theory called Reasoned Equivalence (RE), consisting of If there is Probative Legality (PL) and Legitimacy of the Evidence (LE), it is defined as logically equivalent to Probatory Freedom (PF), therefore, it is not possible to exclude the Evidence (EV), represented under the following formula:

Theoretical construction developed to solve a scientific problem
$RE = PL + LE \Leftrightarrow PF \neq EV$

Chart 1. Proposed formula for a theory of reasoned equivalence. Source [17].

Indicates that, in light of the proposed reasoned equivalence theory, it will positively impact the sphere of criminal procedural law, given that, among other things, the judge must act and adhere to the framework in the objective assessment of the evidentiary means alleged in safeguarding the interests of the intervening parties in all judgment. Otherwise, the court decision could be counterattacked in appeal (superior court), even in administrative headquarters for inexcusable error.

In such a way that the legal principle of non-exclusion from criminal evidence operates as a proposition that is partially true and partially false or partially indeterminate for the operators of justice.

So evidentiary law is a branch of procedural science in criminal matters for a category of the population that may be convenient; but also negative for another part of the intervening parties in the criminal trials.

Everything will depend on the role to be played, either as a defense or accuser party into procedures record trial.

It is limited that, under being the dynamic neutrosophic degrees, that is, they are not static, they can continually change over time around hidden parameters that influence each other.

Thus, in all societies we find neutrosophic degrees of positive (T), indeterminate, or neutral (I) and negative (F) attributes, therefore, we could say that in any society, we have the following neutrosophic degrees. The degrees T, I, F are independent concerning each other [18].

(Ti, Ii, Fi) - inequality, (Tu, Iu, Fu) - dissatisfaction, (Tc, Ic, Fc) -contradiction, (Tw, Iw, Fw), error of law, among others, unlike Auguste Comte in Smarandache (who coined the term “perfect sociology”, given that we are people imperfect by nature and to that extent, we can make the mistake of fact and law [19].

On the other hand, a line-by-line analysis was carried out, which led to an important theoretical approach by correlating the context in which the central category (criminal evidence) is found and the subcategories (declaration of the accused, interrogation or examination, cross-examination) or counter-examination, judicial inspection, recognition of things, people and places and the valuation system), without prioritizing them, given the absence of hierarchies: axial codification emerged [17].

Even when the axial coding is not predominant because the process of establishing relationships was executed against the central category; but certainly plausible; by establishing a flexible class of the subcategories described above with the properties and dimensions around a category taken as a transversal axis (criminal evidence), a scheme was obtained that facilitates the understanding of the phenomena that provide a procedural process to configure the category central.

This finding demanded to describe previously in what legal context the evidentiary function is developed at present with a greater emphasis in the generalization this time from the lens of the probation articulated to the system of evaluation of the criminal test based on the theory of reasoned equivalence proposal.

5. Evaluative indicators of the principles in the evidence

The legal means of evidence constitute the regulated instruments provided by the national legislator; they identify the indicators that represent the conviction of the alleged allegations on which the oral litigation of the exercise of probative law is based. Indicators are the key element for determining the truth in all criminal proceedings. Chart 2 shows the evaluative indicators obtained in the activity.

N°	Evaluative indicator
11	Legality and legitimacy of the documentary evidence
12	Freedom from testimonial evidence
13	Relevance of expert evidence
14	Unlawfulness of other evidence
15	Exclusion from legal evidence

Chart 2. Evidence indicators.

After the analysis of the information codified in the Organic Integral Criminal Code of Ecuador [20], the coefficients of knowledge, argumentation, and jurisdiction were determined, in the evaluation of the judge, the legal proposal provided by five criteria (Strongly Agree, Agree, Little agree, Disagree, Did not answer), applied to five variables based on the Likert scale.

For data collection, a Likert questionnaire is designed. This type of questionnaire is described as the method that uses an instrument or form, intended to obtain answers about the problem under study and that the researched or consulted person completes by himself [21].

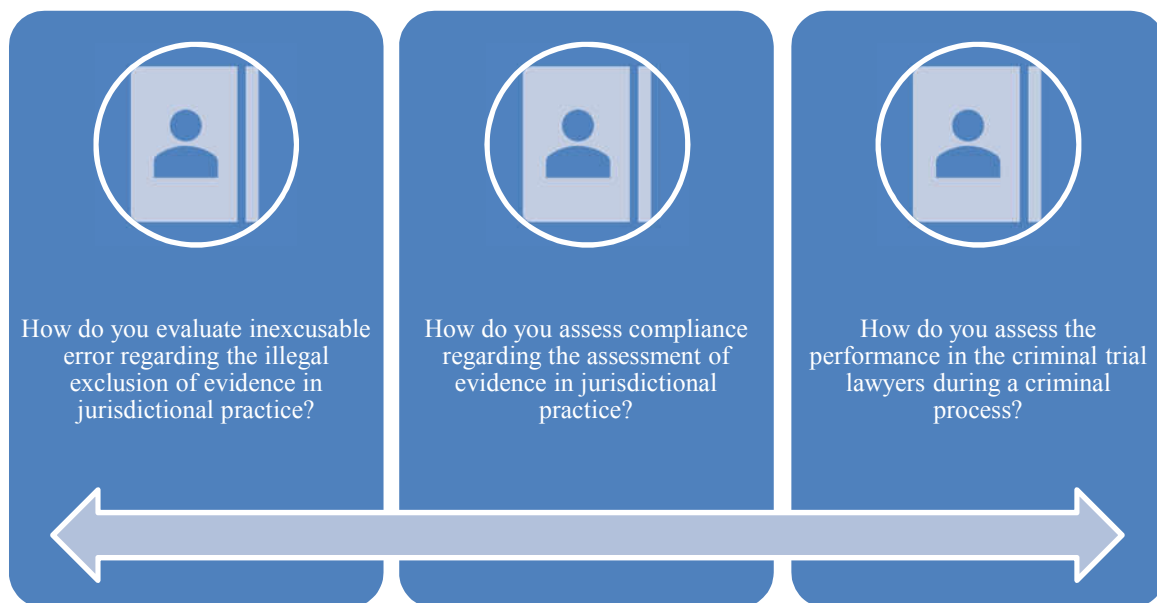


Chart 3. Questions asked internet to key informants

Most respondents who reached ninety-nine percent strongly agree that inexcusable error is not conceptualized in the Organic Code of the Judicial Function, but is generically incorporated into very serious offenses but without a clear definition of its meaning, for what has been done extensive interpretations of the norm, causing the rights of judicial officials to be violated.

In attention to the intentional sample population, it was made up of 60 criminal lawyers, who without the need to physically gather them, collaborated with the information from the data provided. Meanwhile, the data collection instrument fell into a guide for observation and recording of debate and dialogue structured with 3 questions concerning the sensitive experience of the professional exercise of key informants to reveal information. Therefore, the participant-observation allowed to check the phenomenon that is in front of the view, with the concern of avoiding and preventing the observation errors that could alter the perception of a phenomenon or the correct expression of it. In this sense, the observer is distinguished from the key informant since the latter does not attempt to reach the diagnosis [22].

The analysis carried out and expressed allowed determining the values of the cut-off point of the categories. These values were related to the step value category (N-P) of each expressed variable.

In the analysis of the results of the assessment of the contribution of the model, it was found that all items were evaluated as Strongly Agree, Agree or Disagree, as shown in chart 3.

Capita	C1	C2	C3	C4	C5
	Strongly Agree	Agree	Little agree	Disagree	Did not answer
1	99%	1%	0%	0%	0%
2	10%	15%	75%	0%	0%
3	44%	16%	40%	0%	0%

Chart 3. Refined Neutrosophic [14]. Result of the observation guide instrument applied to key informants (collaborated criminal lawyers) to evaluate the proposal made.

Among the criteria issued by the experts consulted using the Delphi methodology [23, 24], the following elements prevail:

- The indicators for measuring the exercise of evidentiary law to assess jurisdictional practice were considered correct.
- The fulfillment of the evaluative indicators of the jurisdictional practice “Little agreement” being considered under its development by the repetition in the exclusion of the evidence on the part of the judge when the time of their evacuation arrives at the procedural stage of evaluation and preparatory trial.
- The growth of the indicator criminal trial lawyers during a criminal process is considered practically between “Strongly agree” and “Little agree”.

In addition to the favorable criteria on the proposed model, the following suggestions and recommendations were issued by the experts:

It must be considered that, although the level obtained in the evaluative indicator of jurisdictional practice must prevail the guiding principle of probation, as long as the evidence is legal and legitimate by establishing that there is

no place to exclude criminal evidence based on the principle of discretion of the judge for the responsibility that he carries while avoiding abuses in the administration of justice.

It is important to indicate considerations on the contribution that is made to the research, given that, among other things, from a positive point of view, [25] inclined to reflect on the evidence system in general, it contributes ideology to be able to affirm that contemporary theories or standards of evidence are not fully met. In other words, it argues that: contemporary law not only programs its forms of production but also its substantial contents, linking them normatively with constitutionally recognized principles and values [26].

6. Conclusions

As a corollary, armed with the elements that link the various critiques to the content of each test, the theory of reasoned equivalence proposed, bet that judicial operators, who sometimes act within surrealism, are inserted to the extent that for some reason consider that the object of knowledge is separate from the subject that knows or what is the same that the knowledge of the object differs from the subject to know. The judge cannot ex officio promote any evidence. However, his faculty should endow it with such a possibility, at least with the limitations of the case since it is at the expense of his investiture.

Therefore, it should be noted that for the simple fact, that there is currently individual background to this movement surrealist, sometimes negatively, to demonstrate a particular disposition of the spirit that plunges into the depths of the real, seeking a basis to affirm its faculty to the detriment of judicial activism. To that extent, the pretext of surrealism will be useful for the discovery of its essence -as it is intended to demonstrate-, its permanent updating and way of assuming the reality [27].

In such a way that knowledge is an exact reproduction of reality, and if it is totally or partially unknown, it is because elements of judgment are missing, or simply the evidence was disturbed. Considering, whether or not it is conducive or apt, in the abstract, to be able to prove a fact or legal act, it is a point of law, because it deals with the application of the legal means that regulate the test in a particular case and therefore, the concept of the court of appeal may be attacked in cassation by mistake of law if it is considered wrong.

This is important because in some countries certain proof can and other proof cannot be used as evidence in criminal cases, meaning that the rules on the admissibility of evidence and the high standard of proof required in criminal proceedings it necessarily needs to apply in this respect. In that case, juridically that would be an inexcusable error, and ethically, an illegal judge decision.

Given the strict rules on the admissibility of evidence and the high standard of proof required in the criminal justice systems of Ecuador, including, as appropriate, through legislative changes, that would facilitate the use of such evidence in criminal proceedings.

Finally, in an attempt to contribute, the proposed theory of equivalence bets on a greater and better administration of justice. It will tend to make criminal trials more expeditious, but above all mayor transparency when it comes time to acquit the accused or on the contrary condemn those whom injustice deserves it.

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